CALDER'S
ARITHMETIC
PART.
I. & II.

4/6.

Reduction has been divided into two parts, the simpler operations being placed before the Compound Rules, and the more difficult part after them. In treating this subject, care has been taken to avoid the common mode of bringing too rapidly before the pupil examples of various kinds; and the object has been to lead him very gradually through all the successive stages and difficulties which that Rule presents.

I have also endeavoured to introduce a gradual knowledge of the signs of Addition, Subtraction, &c. For though it is easy for a pupil to learn them one at once, and apply them by degrees; yet I have found, that when a boy, on entering Fractions, has to learn the use of the signs all at once, he often becomes confused, and is consequently much hindered in that part of his work, where a ready use of the signs is so indispensable.

After every two or three Rules, miscellaneous practical examples have been introduced, sufficiently easy, it is hoped, to elicit the power of applying the Rules which have been learned; a power of which there is in most pupils such a marked deficiency. A few questions, also, on the principles which have been explained, will initiate, at least inquiring pupils, into the habit of self-examination in the subjects which they have professed to learn.

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Grammar, School, Chesterfield,
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CONTENTS.

ARITHMETIC.

NOTATION AND NUMERATION.

1. When several things of the same kind are placed before the eye, even of a child, and a person either takes some of them away, or puts some more to them, the child perceives that there has taken place an alteration,—but not in the *kind* of the articles. Hence he first gets his idea of *number*, and asks the question, How many things?

Or, if he divides a heap of marbles among his companions, giving two to each till the heap is all gone, the question comes into his mind—How many times round can I go?

Now the science which will enable him to answer the questions of *How many things?* and *How many times?* is ARITHMETIC.

His ideas of numbers are as yet expressed only in words; but he afterwards learns that these numbers can be written shorter in *figures*. Now the art of changing these words into figures which have the same meaning is called NOTATION, and that of reading figures in their proper words is called NUMERATION.

The following are the figures or digits used in Arithmetic, and by means of which all numbers whatever may be expressed.

	0,	and it is thus read in words,		*Nought* or *cipher*,
	1,	,,	,,	*One*,
	2,	,,	,,	*Two*,
	3,	,,	,,	*Three*,
	4,	,,	,,	*Four*,
	5,	,,	,,	*Five*,
(A)	6,	,,	,,	*Six*,
	7,	,,	,,	*Seven*,
	8,	,,	,,	*Eight*,
	9,	,,	,,	*Nine*,

being altogether TEN in number. *

2. The reason why there should be used only ten different figures is, because there are ten fingers on the hand; and these have been early used for counting: thus, beginning with the thumb of one hand, and going through both hands, a boy finds that he can count up to *ten;* but when he comes to the last or tenth finger, and he wishes to count any more, he must make some mark or note to shew that he has been *once* round. This is done by taking the figure 1, and placing a 0 at its right hand, writing it thus, 10, where the figure 1 which formerly stood for one only, now stands for one round of the fingers, or *one ten,* which we call *Ten.*

If now I begin to go over the hands again, and thus make *two* rounds, or two tens, which are called *Twenty,* I ought to write a figure 2 where I before had 1, and write **20**: If I take 3 rounds, or 3 tens, I must write 3 instead of 2, and say **30**, or Thirty: So also

4 rounds give 4 tens, or **40**, called *Forty.*
5 rounds give 5 tens, or **50**, called *Fifty.*

* The numbers 1, 3, 5, 7, &c., are called *odd*, and 2, 4, 6, 8, &c., are called *even* numbers.

(B)	6 rounds give	6 tens, or	**60**, called *Sixty*.
	7 rounds give	7 tens, or	**70**, called *Seventy*.
	8 rounds give	8 tens, or	**80**, called *Eighty*.
	9 rounds give	9 tens, or	**90**, called *Ninety*.
	10 rounds give	10 tens, or	**100**, which has now a new name

called a hundred, or ONE *Hundred*.

Exs. 1. Write down in figures, Forty, Seventy, Sixty, Thirty, Ninety, Twenty, Fifty, Ten, Eighty, One hundred.

Write down in words, 20, 60, 90, 100, 50, 30, 80, 70, 40.

OBS. The pupil should be perfectly ready at reading and writing the above examples, before he advances a step further.

We now find that when two figures are put at the end of the 1, as in **100**, this figure is called ONE *hundred;* so, if we write 2 instead of 1, and have

	200,	we shall call it	*Two hundred,*
	300,	"	*Three hundred,*
	400,	"	*Four hundred,*
	500,	"	*Five hundred,*
(C)	600,	"	*Six hundred,*
	700,	"	*Seven hundred,*
	800,	"	*Eight hundred,*
	900,	"	*Nine hundred,*
	1000,	"	*Ten hundred,*

which is called ONE *Thousand.*

Exs. 2. Write down in figures, Three Hundred, Eight Hundred, Seven Hundred, One Hundred, Ten Hundred, or One Thousand, Four Hundred, Two Hundred, Five Hundred.

Write down in words, 600, 800, 300, 100, 1000, 900, 400, 700, 200, 500.

3. We now learn that when *three* figures are placed after the 1, this figure 1 is called *One Thousand;* and we shall find it convenient to put a comma before these three figures, and write 1,000. When, therefore, we see three figures at the end of the 1, and a comma placed, we shall easily remember to read in words, ONE *Thousand.*

So, if there are three figures after the figure 2, we shall write

	2,000,	which is read	*Two thousand,*
and in like manner,	3,000,	„	*Three thousand,*
	4,000,	„	*Four thousand,*
	5,000,	„	*Five thousand,*
(D)	6,000,	„	*Six thousand,*
	7,000,	„	*Seven thousand,*
	8,000,	„	*Eight thousand,*
	9,000,	„	*Nine thousand,*
	10,000,	„	*Ten thousand.*

and in like manner, changing 10 into 20, 30, &c., up to 100, we have

	20,000,	which is read	*Twenty thousand,*
	30,000,	„	*Thirty thousand,*
	&c.		&c.
(E)	60,000,	„	*Sixty thousand,*
	&c.		&c.
	90,000,	„	*Ninety thousand,*
	100,000,	„	*One hundred thousand.*

And still further, changing 100 to 200, 300, &c., up to 1,000, we have

	200,000,	which is read	*Two hundred thousand,*
	300,000,	„	*Three hundred thousand,*
	&c.	„	&c.
(F)	600,000,	„	*Six hundred thousand,*
	700,000,	„	*Seven hundred thousand,*
	&c.		&c.
	1,000,000,	„	*One thousand thousand,*

and this is called by a new name, ONE *Million.*

Exs. 3. Write down in figures, Five Thousand, Nine Thousand, Thirty Thousand, Three Hundred Thousand, Eighty Thousand, Seventy Thousand, Seven Hundred Thousand, One Million.

Write in words, 4,000, 80,000, 600,000, 800,000, 1,000, 70,000, 1,000,000.

4. Having now *six* figures after the 1, and a comma placed after every three figures, (counting from the right

hand to the left,) we must remember that all figures to the left of these six right-hand figures are MILLIONS; we shall thus have

	2,000,000,	which is read	*Two millions,*
	&c.		&c.
	5,000,000,	„	*Five millions,*
(G)	6,000,000,	„	*Six millions,*
	&c.		&c.
	10,000,000,	„	*Ten millions,*
	&c.		&c.
(H)	60,000,000,	„	*Sixty millions,*
	100,000,000,	„	*One hundred millions,*
	200,000,000,	„	*Two hundred millions,*
	&c.		&c.
(I)	600,000,000,	„	*Six hundred millions,*
	&c.		&c.
	1,000,000,000,	„	*One thousand millions,*
	&c.		&c.
(K)	6,000,000,000,	„	*Six Thousand Millions.* *

Exs. 4. Write down in figures, Four Millions, Nine Millions, Seventy Millions, One Hundred Millions, Three Hundred Millions, Eight Hundred Millions.

Write down in words, 40,000,000, 8,000,000, 60,000,000, 2,000,000, 50,000,000, 80,000,000, 600,000,000, 1,000,000,000.

5. We have now shown how to read in words any single figure, either by itself, or when followed by any number of figures; and we will now write out in the form of a table the results which we have just been proving.

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* We do not often require numbers higher than these. We can, however, read as follows.

	10,000,000,000,	that is	*Ten thousand millions,*
	&c.		&c.
(L)	60,000,000,000,	„	*Sixty thousand millions,*
	&c.		&c.
	100,000,000,000,	„	*One hundred thousand millions,*
	&c.		&c.
(M)	600,000,000,000,	„	*Six hundred thousand millions,*
	&c.		&c.
	1,000,000,000,000,	„	*One thousand thousand millions,*
		or,	*One million millions,* which is called

ONE BILLION. If there were six more figures to the right of the 1, it would become One *Trillion*; if 6 more, a *Quadrillion*; and so on, the highest separate name being a *Nonillion*.

Collecting a single line from each of the preceding groups of figures, viz. those marked A, B, C, D, E, F, G, H, I, K, L, M, and placing them so that the last figures in each row will stand under each other, we have

A.	6	which is	*Six.*
B.	60	"	*Sixty.*
C.	600	"	*Six Hundred.*
D.	6,000	"	*Six Thousand.*
E.	60,000	"	*Sixty Thousand.*
F.	600,000	"	*Six Hundred Thousand.*
G.	6,000,000	"	*Six Millions.*
H.	60,000,000	"	*Sixty Millions.*
I.	600,000,000	"	*Six Hundred Millions.*
K.	6,000,000,000	"	*Six Thousand Millions.*
L.	60,000,000,000	"	*Sixty Thousand Millions.*
M.	600,000,000,000	"	*Six Hundred Thousand Millions.*

Units.
Tens.
Hundreds.
Thousands.
Tens of Thousands.
Hundreds of Thousands.
Millions.
Tens of Millions.
Hundreds of Millions.
Thousands of Millions.
Tens of Thousands of Millions.
Hundreds of Thousands of Millions.
&c.

6. Underneath the last line are written the names of all the values which the 6 has in all its different positions; and if these are learnt by the pupil, the value of any one figure which he may desire to know will be found immediately. Thus, if I have the number 500000, without the commas to help me to see *at once* what the 5 stands for, I shall begin with the right-hand figure, and proceed to the left, pointing to the figures one after another with the finger or pencil, and shall say, *units, tens, hundreds, thousands,* &c. till I come to the 5, and I shall then find that I

have come to *hundreds of thousands;* hence the 5 in the above number stands for *five hundred thousands.*

7. We have now to show how all the intermediate numbers, between those which we have explained, are to be written both in figures and in words.

In (1) we went up as far as Ten or 10; and then went to Twenty or 20; the intermediate numbers are

In words	In figures
Eleven	11
Twelve	12
Thirteen	13
Fourteen	14
Fifteen	15
Sixteen	16
Seventeen	17
Eighteen	18
Nineteen	19
Twenty	20

Take now any one of these numbers, as 16, and it will be observed that it is composed of ten and six; that is, the 1 stands for *ten,* as in the figures 10, and the 6 is added to it; so 17 means ten and seven together; 19 means ten and nine together; and so on.

In like manner we can fill up all the places between 20 and 30;

Thus,	24	stands for	20 and 4,	or *Twenty-four.*
	27	„	20 and 7,	or *Twenty-seven.*
so also,	32	„	30 and 2,	or *Thirty-two.*
	45	„	40 and 5,	or *Forty-five.*
	49	„	40 and 9,	or *Forty-nine.*

And the same method is used for expressing the numbers between 50 and 60, between 60 and 70, &c.

Exs. 5. Write in figures, Fourteen, Twenty-Two, Sixty-one, Seventy-Seven, Eighty-Five, Ninety-Three, Forty-Eight, Fifty-One, Thirty-Four, Ninety-one, Nineteen.

Write in words, 28, 37, 45, 89, 73, 64, 58, 17, 21, 88, 33, 99.

8. Proceeding beyond 100, we fill up all the numbers between 100 and 200, by writing all the numbers from 1 up to 99, instead of one or both of the ciphers in 100.

Thus, if I have to write *one hundred* and *fifty-seven*, I first put down 1 for the *one hundred*, and then, instead of the two ciphers, I write *fifty-seven* in figures, that is, 57; so that it becomes 157. Similarly, *one hundred* and *seventy-nine* is written 179; and *one hundred* and *eight* is 108, where I see that the number eight, or 8, takes the place of only the *latter* of the two ciphers; but in one hundred and fifty, or 150, the 50, being 5 tens, stands as a 5 in the tens' place, and the latter cipher is unaltered.

The numbers between 200 and 300, between 300 and 400, &c., are filled up in exactly the same way; so that if a pupil has learnt thoroughly what has been written above, he will see at once that

375	stands for	*Three hundred and seventy-five.*
401	"	*Four hundred and one.*
517	"	*Five hundred and seventeen.*
833	"	*Eight hundred and thirty-three.*
760	"	*Seven hundred and sixty.*
999	"	*Nine hundred and ninety-nine.*

Having now learnt how to write in words and in figures all numbers from 1 to 1,000, the pupil will easily see that all the numbers which he has just been learning will fill up the space from 1,000 to 2,000, from 2,000 to 3,000, &c.

so that	1324	stands for	*One thousand three hundred and twenty four.*
	2075	"	*Two thousand no hundreds and seventy-five,* or *two thousand and seventy-five.*
	3104	"	*Three thousand one hundred and four.*

Exs. 6. Write in figures, Five Hundred and Seven, Eighteen Hundred and Nineteen; Four Thousand and Seventeen; Nine Hundred and Twenty, Eight Hundred and Eleven; Nine Thousand Four Hundred and Sixteen, Two Thousand and Four.

Write in words, 302, 427, 109, 704, 1845, 1900, 7321, 8907, 1401, 3870, 1044, 9009, 10,000.

9. I need not describe at length the mode of writing all the numbers from 10,000 to 20,000, 30,000, &c.; or from 100,000 to 1,000,000, &c. If the pupil will bear in mind what was said in (3) and (4), that all millions have *six* figures after them, and all thousands *three* figures after, he will find little difficulty in writing down any proposed number. Also, when learning to write down long numbers, he should place nine dots, as in the margin, to guide him where to put the three different terms *millions, thousands,* and numbers below 1000.

Mill^ns	Thous^s	
...	...	...
345,	126,	327.

Thus, if I have to write down 345 millions, 126 thousands, and 327, I must place the 345 under the first three dots, the 126 under the next three, and the 327 under the last.

I see here that all the 9 places are filled up by the figures which I had to put down; but if the numbers given had been such as not to occupy all the places, then the empty places between the first and last figures must be filled with ciphers. For instance, if a beginner has to write 25 millions, 7 thousand, and 29, he should first write as follows:

	...	...	...
first write as follows:	25,	7,	29.
and then filling up the empty places, he will have	25,	007,	029.

And he will now see the use of the ciphers, namely, to keep all the other figures in the places which we have shewn to be the proper ones.

Obs.—I do not place a 0 under the last dot to the left, because there are no figures to the left of it, to be kept in their proper places.

Exs. 7. Write down in figures, Fifty Thousand and Six, One Hundred and Seventy Thousand and Eighteen, Four Hundred and Five Millions Thirty-Nine Thousand and One, Three Thousand and Five Millions Six Hundred and Nine, One Million and Forty-Three.

Write down in words, 700,401, 1,400,906, 40,010, 80,040,017, 400,401,090, 36,011, 3,245,068,018.*

ADDITION.

10. To Add is to collect together two or more numbers into one sum.

In order to add together any numbers, we must collect, separately, all those numbers which are of the same kind; thus, units must be added to units, tens to tens, hundreds to hundreds, and so on.

When, therefore, we have to find the sum of any numbers, we must so place them under one another, that all the figures in the units' places may be in an upright row; then the tens, hundreds, thousands, &c., if there be any, will also be arranged in upright rows.

11. To help the learner to add together any two numbers, each less than 10, I insert the following Addition Table.

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

* The figures which we have been describing are *Arabic* Numerals.

The Roman Numerals were I for 1, V for 5, X for 10, L for 50, C for 100, D for 500, M for 1000.

Any one of these placed to the *right* of a larger numeral was to be *added* to it; thus VI represented 5 and 1 or 6; XII stood for 12; and CLXX for 170.

But a numeral placed to the *left* of a larger numeral was to be *subtracted* from it; thus IV stood for 4; XL for 40.

Sometimes, also, IƆ was written for 500, and CIƆ for 1000.

It may be learnt or read thus. Place a finger of the left-hand on any one of the figures in the left-hand row; and as you move your finger to the right, add to this figure the numbers in the highest row one after another. The amounts will be under the finger at every step. Thus, pointing to 4 with the left-hand, and to the 1, 2, 3, &c., of the top line with the right-hand, and then moving both hands to the right, we read 4 and 1 make 5, 4 and 2 make 6, 4 and 3 make 7; and so on, to the end of that line; and I see that the amounts 5, 6, 7, &c., are exactly under the 1, 2, 3, &c., of the top line, and in the same row as the left-hand 4.

If now I wish to find from this table the sum of any two numbers, each less than 10, as 7 and 8, I point with the left-hand to the 7, in the left-hand row, and with the right-hand to the 8 in the top row; then moving the right-hand straight down till it comes opposite the left-hand, I find I come to the number 15. This 15 is the *sum* of the numbers 7 and 8.

12. We have seen in NUMERATION, that the figure 1 in the second place from the right-hand counts as 10; therefore, to add 10 to any number under 10, we have only to place the figure 1 to the left of that number; thus, if I have to add 10 to 7, I merely write 17. So also, to add 10 to a number greater than 10, as to 46, I add 1 to the figure in the *tens'* place, so that 46 and 10 make 56. By the same method 235 and 10 make 245; 1427 and 10 make 1437. In like manner, to add 100, I increase the figure in the *hundreds'* place by 1, so that 245 and 100 make 345; 3156 and 100 make 3256.

We shall now have no difficulty in adding together two numbers, one of which is more than 10, and the other less

than 10, as 16 and 3. For 16 is the same as 10 and 6. I therefore add the 3 to the 6, making 9; and keeping the 1 in the units' place, make 19 in all.

$$\begin{array}{r} 16 \\ 3 \\ \hline 19 \end{array}$$

So also, to add 16 and 7, I first add the 7 to the 6, making 13, and write down the 3; and for the 10 in this number 13, I put 1 more in the tens' place, and make in all 23.

$$\begin{array}{r} 16 \\ 7 \\ \hline 23 \end{array}$$

In like manner, if I have to add 28 and 7, I shall first add the 7 to the 8, making 15, and put the 5 in the units' place; and for the 10 in this 15, I add 1 to the 2 in the tens' place, making 35.

$$\begin{array}{r} 28 \\ 7 \\ \hline 35 \end{array}$$

And so I might add any two such numbers, as 36 and 7, making 43; 46 and 7, making 53; 146 and 7 making 153; 248 and 8, making 256.

Exs. 8. Write down the sums of 16 and 10, 26 and 10, 13 and 5, 18 and 9, 26 and 7, 45 and 8, 78 and 4, 89 and 1, 101 and 8, 104 and 7, 156 and 8, 345 and 9, 426 and 7.

13. Now let it be required to find the sum of 2548, 8027, 9, and 765.

Arranging the numbers according to the directions given above, so that all the figures in the units' places may be in an upright row, they stand thus:

$$\begin{array}{r} 2548 \\ 8027 \\ 9 \\ 765 \\ \hline 11349 \end{array}$$

Beginning at the right-hand or units' row, and adding, I say;—5 and 9 are 14, 14 and 7 are 21, 21 and 8 are 29, that is, 2 tens and 9 units; I place the 9 units under the row of units, and carry the 2 tens to the next row which consists of tens. I now add this second row, just as I did the first; it amounts to 12 tens, and with the two tens carried, it is 14 tens, (or 140,) that is, 1 hundred and 4 tens; I place the 4 under the row of tens, and carry the 1 hundred to the next row which con-

sists of hundreds. The third row, with the 1 carried, amounts to 13 hundreds, (or 1,300), that is, 1 thousand and 3 hundreds; I put down the 3 under the row of hundreds, and carry the 1 thousand to the fourth row: this amounts to 11 thousands; and as there are no more rows to add, I put down the whole of the 11. The complete answer is, Eleven thousand three hundred and forty-nine. And so we might have added any number of rows. Hence we have the following Rule for SIMPLE ADDITION:

RULE. Arrange the numbers to be added under one another, so that the figures in the units' places may be in an upright row, and draw a line under the whole.

Add up the first row to the right, and of the amount just found place the right-hand or units' figure under this row, and carry the other figure, or figures, if there be any, to the second row. Find the sum of this second row, adding in the figure carried from the first row; put down the right-hand figure, and carry the remaining ones as before. Proceed in like manner to the last row; and when its sum is found, place the whole of it to the left of the figures already set down.

OBS. The operation of Addition is sometimes denoted by the sign $(+)$ *plus*, which, placed between two numbers, shows that they are to be added together. Also, the sign $(=)$ *equal to*, placed between any two quantities, signifies that they are equal. Thus $4 + 5 = 9$, is read 4 *plus* 5 *equals* 9, or 4 added to 5 makes 9. So also, $4 + 5 + 3 + 7 = 19$; or the *sum* of 4, 5, 3, and 7, is 19.

EXS. 9. Find the value of

1. $643 + 879 + 245 + 134 + 851 + 405 + 137$.
2. $145 + 672 + 834 + 1000 + 987 + 451 + 1687$.

3. 1144 + 328 + 456 + 987 + 101 + 376 + 451.
4. 4876 + 3459 + 1111 + 4321 + 6897 + 14562.
5. 8976 + 3178 + 9017 + 5632 + 14587 + 8765.
6. 4301 + 9872 + 4632 + 1829 + 5437 + 684 + 9187.
7. 8743 + 1986 + 4530 + 12875 + 1493 + 6421.
8. 1879 + 8431 + 9645 + 4287 + 1673 + 7482,
9. 8735 + 6419 + 4327 + 9180 + 14327 + 6875.
10. 3456 + 875 + 10010 + 7435 + 8962 + 67.
11. 8432 + 98765 + 20101 + 3476 + 823 + 10479.
12. 658 + 1034 + 98710 + 3279 + 14826 + 4113.
13. 84291 + 103456 + 8732 + 91011 + 178452 + 74379.
14. 110456 + 83019 + 75146 + 238108 + 14679 + 8001.
15. 148765 + 75832 + 64101 + 75 + 80015 + 9873.
16. 143856 + 28739 + 41032 + 999 + 91458 + 100000 + 87532.
17. 457890 + 32576 + 987542 + 17639 + 4010 + 101010 + 67348.
18. 111111 + 43210 + 87641 + 12479 + 71 + 375 + 118934.
19. 843217 + 641839 + 4813756 + 5204187 + 28756 + 4315821 + 867509.
20. 43201 + 384917 + 1007 + 138589 + 710234 + 86549 + 684305.
21. 148302 + 7649 + 45831 + 776789 + 145384 + 784591 + 638414.
22. 487643 + 1897421 + 684579 + 4458321 + 9237548 + 8459320.
23. 876410 + 1984321 + 765899 + 4101873 + 600750 + 9999.
24. 4865320 + 6421987 + 468359 + 146721 + 1098638 + 1016.
25. 687532 + 4591408 + 7432017 + 5689432 + 874310 + 189 + 6345681.
26. 875439 + 8674392 + 5108469 + 83167 + 410509 + 1673245 + 587410.
27. 5681094 + 6138765 + 494587 + 238654 + 1647829 + 30178 + 472.
28. 784586 + 87532 + 1049761 + 853742 + 945681 + 410392 + 7684910.
29. 487653 + 768923 + 9898989 + 445671 + 830187 + 941245 + 7631408.
30. 456786 + 2389124 + 837540 + 975218 + 111456 + 832075 + 1410987.
31. 8740192 + 1432786 + 910437 + 64985 + 87563 + 11468492 + 753864.
32. 4132875 + 6894360 + 5119387 + 7256918 + 14071728 + 1271449.
33. 489735 + 2010387 + 5897654 + 319771 + 864987 + 410684 + 34586.
34. 8764591 + 143286 + 375 + 914268 + 4372145 + 896451 + 98817532.
35. 348765 + 1487659 + 2018437 + 498765 + 14329 + 87642 + 1073496.
36. 4368291 + 78543 + 689201 + 45763 + 75 + 101789 + 2345682.

SIMPLE SUBTRACTION.

14. To SUBTRACT a less number from a greater is to find how many more units there are in the greater number than in the less; or, to find what number added to the less will make the greater. This number is called the *difference.*

First, I must learn to find the difference between two numbers, whereof one is less than 10, and the other does not exceed it by 10; as for instance between the numbers 9 and 15. For this purpose the Addition Table may be used as a Subtraction Table; thus:—I find the smaller of the two numbers in the left-hand row, and move my finger to the right, till I come to the larger one; I then look to the top of the upright row in which my finger is, and there I find the difference. For instance, taking the numbers 9 and 15, I look for 9 in the left-hand row, and move to the right till I come to 15: at the top of the upright row that my finger now touches, I find 6: this is the difference between the 9 and 15.

15. I now can find the difference between any two numbers. Thus, let it be required to find the difference between 43854 and 2628.

Placing the numbers so that the units' figures shall be under each other, I commence as follows;

$$\begin{array}{r} 43854 \\ 2628 \\ \hline 41226 \\ \hline\hline \end{array}$$

8 from 4, – it cannot be taken; I therefore observe that the 54 in the top line is the same as 40 + 14, or 4 tens and 14; and I subtract the 8 from this 14, finding a remainder 6,

which I place underneath. I now remember that I have no longer 50 in the upper line, but 40; that is, the figure in the tens' place must not be counted as 5, but as 4. I therefore take the 2 tens in the lower line from the 4 tens, and have 2 tens as the difference, which I put down. So, also, 6 hundreds from 8 hundreds gives 2 hundreds, 2 thousands from 3 thousands gives 1 thousand; and since there are no tens of thousands in the lower line to be taken from the 4 tens of thousands in the upper line, the 4 must be brought down as it is. The whole remainder is 4 tens of thousands, 1 thousand, 2 hundreds, 2 tens, and 6, or forty-one thousand two hundred and twenty-six. (41226.)

16. In performing the first subtraction of the above example, because I could not take the 8 from the 4 in the units' place, I added 10 to the 4, or, as it is often called, I *borrowed* 10, and counted the figure 5 in the tens' place as 4, and then subtracted the 2 below it from the 4; but the common way, after having added this 10 in any subtraction, is to leave the figure in the top line of the next place unaltered, but add 1 to the figure under it, and then subtract. Thus, I add 1 to the 2 in the lower line, and then say, 3 from 5, which gives me the difference 2, just as before.

Hence, when it is required to find the difference of two numbers, we have this

RULE. Place the less number under the greater, so that the figures in the units' places may be under one another. Begin at the right-hand; and, if possible, subtract the lower figure from the upper, placing the difference underneath; but if the lower figure is greater than the upper, add 10 to the upper, and then subtract.

Proceed in the same manner with each pair of figures;

but remember that when in any subtraction you have added 10 to the upper figure, you must, in performing the *next* subtraction, add 1 to the figure in the *lower* line. If there remain any figures in the upper line, from which none are to be subtracted, bring them down in their proper order.

OBS. The operation of Subtraction may also be expressed by the sign (–) *minus*, which placed between two quantities shows that the latter is to be taken from the former. Thus, when I wish to say that if 2 be taken from 7, the remainder is 5, I write $7 - 2 = 5$, which is read 7 minus 2 equals 5; or, the *difference* between 7 and 2 is 5.

Exs. 10. Find the value of

1. 45863 – 21051.
2. 68745 – 61032.
3. 58900 – 48799.
4. 456789 – 123456.
5. 785432 – 694310.
6. 689321 – 278543.
7. 5894101 – 785326.
8. 8497103 – 1456897.
9. 4132987 – 875619.
10. 8976410 – 2301019.
11. 9832145 – 4567891.
12. 7640189 – 178923.
13. 1432897 – 567898.
14. 3456890 – 126901.
15. 2117584 – 456897.
16. 18432745 – 1857106.
17. 68935842 – 1065719.
18. 78568932 – 6458109.
19. 385879276 – 7750987.
20. 107964201 – 6423189.
21. 351068743 – 49969836.
22. 784500000 – 6891019.
23. 89760001 – 4321890.
24. 321047607 – 76498321.

QUESTIONS INVOLVING ADDITION AND SUBTRACTION.

Exs. 11.

1. From the sum of 3276 and 189, take the sum of 375 and 456.
2. From the sum of 14095 and 8376, take the difference between 427 and 999.
3. From the sum of 98763, 3275, and 144, take the difference between 10000 and 1001.
4. To the number 10001 add 8399; from the sum subtract 4376; then add 1879; then subtract both 327 and 185; what will remain?

5. Out of 200 marbles, a boy gives away 20, 30, 40, and 50 to four other boys; how many has he left?
6. If the upper line in a subtraction sum be 3456, and the remainder be 749, what is the lower line?
7. If the lower line be 4896, and the remainder be 960, what is the upper line?
8. From one million I take ninety-nine thousand and nine; to the remainder I add seven thousand and fifteen; what will be the amount?
9. Write down in figures and signs; three hundred and seven, added to one thousand and one, is equal to the difference between ninety-two, and one thousand four hundred.
10. Write down in figures and signs; seventy, plus 18, added to five thousand and four, diminished by seven hundred and nine, amounts to four thousand three hundred and eighty-three.
11. Write in words, $1000 - 457 + 193 - 75 = 661$.
12. Write in words, $3045 + 6208 = 10001 - 748$.
13. The flood took place 2348 years B.C.; how many years from that date to the year 1851 A.D.?
14. How many years from the first year of the fifth century to 1850?

SIMPLE MULTIPLICATION.

17. To MULTIPLY one number by another is to see what the first number amounts to, when repeated as many times as there are units in the second number.

Thus, to multiply 7 by 5 is to repeat the number 7 *five* times, that is, it is $7 + 7 + 7 + 7 + 7$, which by addition we find to be 35; and therefore we say that 5 times 7, or 7 multiplied by 5, equals 35. This result, 35, is called the *product* of 7 and 5; also 7 is called the *multiplicand,* and 5 the *multiplier.*

In like manner by performing successive additions, we might find the product of any two numbers; but it is not convenient to obtain these results by addition, any further than to find the sum of 12 repeated 12 times, or the pro-

duct of 12 by 12. A table which contains in order the products of any two numbers not greater than 12, is called the MULTIPLICATION TABLE, and is as follows:

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

The above is thus formed: the first row to the left contains the numbers 1 to 12. Taking the second row with 2 at the top, and going downwards, we find that the row is formed by adding 2 at every step; thus, 2 and 2, or twice 2, are 4; 4 and 2, or 3 times 2, are 6; 6 and 2, or 4 times 2, are 8; and so on, till we come to 12 times 2, or 24. In like manner the 7th row is found by beginning with 7, and adding 7 successively, which gives the row 7, 14, 21, 28, &c. . . . . . . 84. And so for all the other rows.

And if I wish to find the product of any two numbers, each not exceeding 12, as 7 and 8, I point with the left-

hand to the one number, 7, in the left-hand row, and with the right-hand to the other, 8, in the top row, as in the Addition Table; then moving the right-hand straight down till it comes opposite to the left-hand, I find that I come to the number 56; this is the *product* of the numbers 7 and 8.

18. Now let it be required to multiply any number, as 527, by any number not greater than 12, for instance by 9.

Since $527 = 500 + 20 + 7$, therefore, if I multiply each of these three quantities by 9, the sum of all the products will be the whole product of 527 by 9. I have, then, as the first product, 9 times 7, or 63; as the second, 9 times 20, or 9 times 2 tens, which is 18 tens, or 180; as the third, 9 times 5 hundreds, which gives 45 hundreds or 4500: and, as shown in the margin, the sum of these three products is 4743.

$$\begin{array}{r}527\\ 9\\ \hline 63 = 9 \text{ times } 7\\ 180 = 9 \quad ,, \quad 20\\ 4500 = 9 \quad ,, \quad 500\\ \hline 4743 = 9 \text{ times } 527\\ \hline\hline\end{array}$$

19. But in practice we perform this work in one line, and add the products as we go on, thus; 9 times 7 = 63; put down the 3 units, and carry 6 tens; 9 times 2 tens = 18 tens, and with the 6 carried, is 24 tens, or 2 hundreds and 4 tens; put down the 4 tens, and carry the 2 hundreds; 9 times 5 hundreds = 45 hundreds, and with the 2 hundreds carried = 47 hundreds, or 4 thousands 7 hundreds; put down both these figures, and the result is 4743, as before.

$$\begin{array}{r}527\\ 9\\ \hline 4743\\ \hline\hline\end{array} \qquad \text{(A)}$$

Exs. 12. Form the following products.

I. 3456789 by 2, 3, 4, 5, 6, 7, successively.

II. 6410954 by 3, 4, 5, 6, 7, 8.

III. 1875697 by 4, 5, 6, 7, 8, 9.

IV. 6183249 by 7, 8, 9.

We have already seen in Numeration, that if a cipher be placed at the right-hand of a number, every figure in that number has ten times the value that it had before; that is, to multiply a number by 10, I need only place a cipher after it; so, also, to multiply by 100, I place two ciphers; by 1000, three ciphers; and so on.

20. Again, since to multiply the above number, 527, by 10, I merely add a cipher, making 5270; therefore, to multiply it by 20, which is 2 tens, I multiply by 10 and by 2, that is I add one cipher, and then multiply by 2, making 10540. In like manner to multiply by 40, I add a cipher, and multiply by 4; by 60, I add a cipher, and multiply by 6.

So also, to multiply by 200, I add two ciphers, and then multiply by 2; by 700, add two ciphers, and multiply by 7; by 5000, add three ciphers, and multiply by 5.

The following are examples of the manner in which such multiplications are performed.

$$\begin{array}{r} 3247 \\ 60 \\ \hline 194820 \\ \hline\hline \end{array} \qquad \begin{array}{r} 8295 \\ 700 \\ \hline 5806500 \\ \hline\hline \end{array} \qquad \text{(B)}$$

Exs. 13.

I. Multiply 3456789 by 10, 100, 10000.

II. „ 785632 by 20, 30, 600, 8000.

III. „ 4568301 by 9000, 80, 600, 70000.

21. Let us now see the effect of a cipher in the middle of a multiplier. For example, if I have to multiply 48295 by 1703.

Here, multiplying by the 3, according to (A), I obtain the first line of the product; next I multiply by the 700,

according to (B); next by 1000, by merely adding 3 ciphers to the multiplicand, 48295. By adding these three products, I obtain the whole product: but the work is

	48295				8295	
	1703				1703	
	144885	= prod^t by	3		144885	
(C)	33806500	= „	700		3380650	(D)
	48295000	= „	1000		48295	
	82246385	= „	1703		82246385	

usually written as in (D), where all the ciphers are omitted in writing out the products, except the single one in the second line, which came from our having a cipher in the second place of the multiplier.

In working any sum, then, it will be seen that it is not necessary to write down all the ciphers as in (C), but only to place each succeeding row one place farther to the left than the preceding one; but if there be a cipher in the multiplier, it is to be written down before multiplying by the next figure, and the row next below will be thereby thrown one place more to the left.

Hence, when two numbers are given to be multiplied together, we have this

Rule. Place the smaller number beneath the larger, so that the figures in the units' places may be under one another.

Begin with the right-hand figure in the multiplier, and multiply the units in the multiplicand; put down the right-hand figure of this product, and carry the tens, if there be any. Proceed in like manner through the first row, up to the last figure in the multiplicand, when the whole product must be put down.

Multiply in the same way by each figure in the multiplier, placing the first or right-hand figure of each row under the

second figure of the former row; but if there be a cipher in the multiplier, place it under the second figure of the row last formed, and multiply by the next figure in the multiplier, as usual; and in the succeeding multiplication, if there be one, place the row *two* places to the left. Add together all the rows; the result will be the product of the two numbers.

NOTE. The operation of Multiplication may also be expressed by the sign ($\times$), which placed between two numbers shews that they are to be multiplied together; thus, $7 \times 5 = 35$, which is read, 7 multiplied by 5 equals 35; or, 7 *into* 5 equals 35.

Exs. 14. Express the following words in signs and figures.

1. Seven multiplied by four, plus three, minus eight, is equal to twenty-three.
2. Nineteen added to three hundred and twenty-five, is equal to forty three, multiplied by eight.
3. The sum of forty-five and one hundred and nineteen, is equal to the product of four and forty-one.
4. The difference between one thousand and three hundred and fifty-five, is equal to the product of forty-three and fifteen.

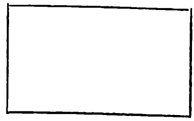
Exs. 15. Form the following products.

I. 6183249 by 11, 12, 13, 14, 15, 16.
II. 8375426 by 124, 347, 645, 809.
III. 45789213 by 653, 842, 976, 1048.
IV. 387940128 by 3456, 7894, 6781, 8592.

1. 4168952×4568.
2. 168942189×13076.
3. 87401329×16849.
4. 53298476×14370.
5. 764109836×4756.
6. 514276439×28309.
7. 8750146081×76112.
8. 458964107×13847.
9. 743286491×68940.
10. 784923607×42816.
11. 814906735×87492.
12. 436892198×57194.
13. 764381479×60879.
14. 9850124657×134806.

22. If I now turn to the Multiplication Table, I find that the number 144 at the right-hand lower corner tells me the number of small squares in the whole square, that is, in a square which has 12 equal parts in each side; so that if each of these parts measures one inch, the whole square will contain 144 squares, each one inch both in length and breadth, or 144 square inches. I thus learn that the number of square inches in the whole square is found by multiplying the number of inches in the length by the number in the breadth.

And if I try this rule on any portion of this square, so that it be enclosed by 4 lines and shaped as in the figure, called an oblong, I shall find it true. For if I take a piece 7 inches long, and 5 broad, I shall find that there are



in it 7 times 5 squares, or 35 square inches; and this number 35 is at the right-hand lower corner of the oblong. If I take a piece 9 inches long, and 5 broad, I shall find 45 squares; so that I can now tell at once how many square inches or feet are contained in any square or oblong, if I know how many inches, or feet, it is in length and breadth, for I have only to multiply the length by the breadth.

If also in each of these small squares a tree were placed, I could tell the number of trees in an oblong clump, by counting the number in length, and the number in breadth, and finding their product.

Exs. 16.

1. A board is 17 inches long, and 10 inches broad, how many squares, an inch each way, are there in its surface?
2. A box is 9 inches long, and 7 inches broad, how many square inches are there in the top and bottom?

3. If the above box be 6 in. high, how many square inches in the sides?
4. In a surface, divided like a chess-board, there are 15 divisions on each side, how many squares in the board?
5. In an oblong plantation, where the trees are planted in regular rows, there are 175 in one side, and 150 in the next side; how many trees are there in all?

SIMPLE DIVISION.

23. To divide one number by another, is to find how often the second number may be subtracted from the first; or, what number multiplied by the second will produce the first. Thus, if it be required to divide 35 by 5, I subtract 5 from 35, till there be either nothing left, or till the number left be less than 5. I find I can subtract the 5 just 7 times, and therefore I say that 35 contains 7 fives, or that, when divided by 5, it gives the answer 7. Here 35 is called the *dividend*, 5 the *divisor*, and 7 the *quotient*.

Also, the operation of Division is represented by the sign ($\div$). Thus, $35 \div 7 = 5$, is read, 35 divided by 7 equals 5; or, the quotient of 35 when divided by 7 is 5.

24. When the number to be divided is not greater than 144, and the divisor not greater than 12, the Multiplication Table may be used as a Division Table. Thus; since I know that if 5 and 8 were to be multiplied together, the result would be the number in the table where the two rows from 5 and 8 meet, viz. 40; therefore, if I make 5 the divisor, and 40 the dividend, then the number 8 standing over the 40 will be the quotient. So, also, if 8 were the divisor, and 72 the dividend, the number 9 at the top of the row over 72 would be the quotient.

But we shall not find all the numbers between 1 and 144 in this Multiplication Table; for instance, if I wish to

divide 43 by 5, and I look in the row beginning with 5, I run my finger along the row till I come to the ***next number below*** 43, that is, 40; and since I find that 8 is at the top of the row, therefore 8 is the quotient; but there are 3 out of the 43 which are not divided; and this work of division which I have just been performing would be thus expressed,—5 in 43 goes 8 times, and 3 over.*

25. But if the dividend consist of four or five places of figures, as 3276, and the divisor be still under 12, as 9, we cannot see at once how many times 9 will produce 3276, but must take several steps to find the result.

Now the 32 in 3276 means 32 *hundreds;* since, then, we know how often 9 is contained in 32, viz. 3 times and 5 over, we know how many times it is contained in 32 *hundreds,* viz. 3 *hundred* times, and 5 *hundred* over. Take 300 times 9, or 2700 from 3276, and we have 576 over, which has not yet been divided. Again, since 570 is 57 *tens,* and 9 in 57 goes 6 times and 3 over, therefore 9 in 57 tens goes 6 tens, or 60 times, and 3 tens, or 30, over. Subtract 60 times 9 from 576, and we have 36 over; and we see that 9 in 36 goes 4 times exactly: therefore the number 3276 has been entirely divided by 9; and the quotient is 300, and 60, and 4, or 364.

$$
\begin{array}{rrl}
 & 9)\ 3276 & \\
 & \underline{2700} & = 300 \text{ times } 9 \\
 & 576 & \\
 & \underline{540} & = \ \ 60 \text{ times } 9 \\
 & 36 & \\
 & \underline{\underline{36}} & = \ \ \ \ 4 \text{ times } 9 \\
\text{or,} & 3276 & = \underline{\underline{364 \text{ times } 9}}
\end{array}
\qquad \text{(E)}
$$

The above may be so arranged as to form a very simple sum, thus:

$$
\begin{array}{r}
9\underline{)\ 3276} \\
\underline{\underline{364}}
\end{array}
$$

And the manner of per-

---

* The pupil will afterwards be shown that if 43 be divided by 5 it is correct to say, that the quotient is eight and *three-fifths*, or, as it is written $8\frac{3}{5}$. See Art. *Fractional Quotient*, in Appendix to Part II. of the Arithmetic.

forming the operation is to say, 9 in 32 goes 3 times and 5 over; put down the 3 under the 2, and take the 5 to the 7, the next figure in the dividend, calling it 57; 9 in 57 goes 6 times and 3 over; put down the 6 under the 7, and take the remaining 3 to the 6, calling it 36; 9 in 36 gives a quotient 4, which put under the 6. Hence, for dividing by a number not greater than 12, we have this

RULE. Place the divisor to the left of the dividend, separating them by a curved line. Take as many figures of the dividend as are necessary to make the number taken at least equal to the divisor: see how often the divisor is contained in this number, and place the quotient under the last of the figures so taken: if there be any remainder, annex it to the next figure of the dividend, and divide as before; but if the remainder and the figure so taken be less than the divisor, place a cipher as quotient, and take another figure and divide. But if there be no remainder, divide the next figure alone, if possible; and if not, put a cipher under it, and take two or more figures if necessary. Proceed in this manner till all the figures in the dividend are taken. If there be any final remainder, place it a little to the right of the quotient.

This method, as shown in the second form of the above example, is called SHORT DIVISION. But as the sum was worked at first, where the subtractions after every division were performed on the paper, instead of in the mind, the process is called LONG DIVISION; and we shall have thus to work almost all examples, where the divisor is greater than 12; the only difference being, that the ciphers might have been omitted in the lines which are subtracted, as was shown in (D) in the multiplication sum in page 22.

Exs. 17. Divide

I. 3849628 by 2, 3, 4, 5, successively.
II. 89764320 by 6, 7, 8, 9.
III. 148763592 by 5, 7, 9, 8.
IV. 4587692015 by 8, 9, 11, 12.

26. The above Rule may also be used for dividing by numbers greater than 12, if they can be exactly split up into two other numbers, each less than 12; as for instance, by 21, which is 7 times 3; by 24, which is 6 times 4. Thus, if we have the number 2476 to be divided by 21, we shall divide, first by one of the numbers, as 7, and then by the other, 3. And to show that this process will bring a correct result, let us suppose we had 2476 apples to be divided into heaps of 21 : if we divide 2476 by 7, we shall have the quotient 353, which gives the number of heaps of 7, and 5 single apples over; and since 3 heaps of 7 will make one heap of 21, therefore, if we now divide these 353 heaps of 7 by 3, we shall have a quotient consisting of heaps of 21; this is 117, and the rem^r 2 represents 2 heaps of 7, which is the same as 14 single apples, and with the former remainder, 5, gives 19 apples rem^s from the whole division. Here we see, that the number remaining from the second division, namely, 2, required to be multiplied by the first divisor, viz. 7, to give the *real* second rem^r, and then it was added to the former remainder. So that when I wish to obtain the complete remainder, 19, from the two partial remainders, 5 and 2, I shall say, $2 \times 7 + 5 = 19$, the true remainder. Therefore, in dividing as in Short Division, by a number which is formed by the multiplication of two numbers under 12, we have, for finding the remainder, this

$$21\left\{\begin{array}{l} 7\underline{)\ 2476} \\ 3\underline{)\ \ 353} \\ \quad\ \ \underline{\underline{117}} \end{array}\right. \begin{array}{l} \\ \left.\begin{array}{l} -\ 5 \\ -\ 2 \end{array}\right\} 19 \text{ rem}^r. \\ \end{array}$$

Rule. Multiply the second remainder by the first divisor, and add in the first remainder. If there be no second remainder, the first remainder will be the true one.

Exs. 18. Divide

I. 37643291 by 14, 16, 18, 21.
II. 874013695 by 25, 36, 56, 81.
III. 172345698 by 72, 49, 121.

Exs. 19. Perform the following operations.

1. 289764235 ÷ 27.	7. 5689743209 ÷ 132.
2. 4568972501 ÷ 33.	8. 78964012385 ÷ 144.
3. 46895017846 ÷ 55.	9. 3456785231 ÷ 84.
4. 7592041863 ÷ 81.	10. 4680397642 ÷ 96.
5. 5697421078 ÷ 121.	11. 58742136847 ÷ 99.
6. 6498721417 ÷ 120.	12. 84397652108 ÷ 108.

27. But since most of the numbers that we have to use as divisors cannot be broken up exactly into two numbers each under 12, we must use the method of (E), or of Long Division, when dividing by any ordinary divisor larger than 12. I will give another Example.

Let it be required to divide 3487906 by 754.

Now as in Art. (25) we showed how to divide 3276 by 9 at several steps, so we must work in this example. Take the 3487 as being the smallest number of the dividend that can be divided by 754; and since there are 3 places after these four figures, we know that this 3487 signifies 3487 *thousands*, and therefore when divided by 754, the quotient will be so many *thousands*.

We have first to see how often 754 goes in 3487; this is nearly the same as seeing how often 7 hundred goes in 34 hundred, or 7 in 34; that is, we neglect the 2 right-

hand figures and then divide. Here 7 in 34 goes 4 times and something over, therefore 700 in 3400 goes 4 times, and something over: but we said that 3487 in this place meant 3487 thousands, therefore 754 in 3487 thousands goes 4 thousands and something over; put this 4000 in the quotient; and that we may see how much is over, subtract from the whole dividend, 4000 times 754, or 3016 thousands, and we have a rem^{r} 471906. Taking this rem^{r}

$$
\begin{array}{rrll}
754) & 3487906 & & \\
 & \underline{3016000} = & 4000 & \text{times the divisor} \\
 & 471906 & & \\
 & \underline{452400} = & 600 & \quad ,, \\
 & 19506 & & \\
 & \underline{15080} = & 20 & \quad ,, \\
 & 4426 & & \\
 & \underline{3770} = & 5 & \quad ,, \\
 & \underline{\underline{656}} & & \\
\end{array}
$$

$$\text{that is, } 3487906 - 656 = \underline{\underline{4625}} \text{ times the divisor.}$$

as a new dividend, and beginning, as before, by dividing into the first four figures 4719, which are so many *hundreds*, we find that 754 in 4719 goes 600 times, and something over: as before, we subtract 600 times 754 from the new dividend, to see how much is over, and we find 19506. Again, dividing by 754, we find that it goes 20 times and something over, and our rem^{r} after subtracting is 4426: here 754 goes 5 times, and something over; and by subtracting again, we have the last rem^{r} 656; and our whole quotient is $4000 + 600 + 20 + 5$, or 4625.

We have here performed the process at full length: but if we put down only those figures which are necessary in the operation, the sum will appear thus.

```
754) 3487906 (4625
     3016
     ----
      4719
      4524
      ----
       1950
       1508
       ----
        4426
        3770
        ----
         656
```

We here see that after each subtraction it is not necessary to bring down to the rem^{r} more than one figure of the dividend, though it sometimes happens, as in Short Division, that the rem^{r} is so small, that even when one figure is brought down, the rem^{r} still is less than the divisor; in this case a cipher must be placed in the quotient, shewing that there has been no division performed, and another figure must be brought down, and the work proceeded with as before.

I insert one more Example which will show that care must be taken when one figure is brought down, and yet no division performed.

Divide 1746549138 by 4587.

```
4587) 1746549138 (380760
      13761
      -----
       37044
       36696
       -----
         34891
         32109
         -----
          27823
          27522
          -----
           3018
```

Here, when, after the second subtraction, the 9 was brought down, the dividend 3489 was smaller than the divisor, and no division could be performed; I therefore put a 0 in the quotient, and brought down another figure, and the divisor then could go into the dividend, 7 times. So, also, after the last subtraction, when 8 was brought down, the number 3018 was too small to be divided, and a 0 was placed in the quotient, and 3018 was left as a remainder.

It seems unnecessary, after going so fully through the above example, to state a separate rule for Long Division,

since the only difference in working Short and Long Division is, that the subtractions necessary to find the rem^{rs}, after every division, are in *Long* Division performed on the paper, whereas in *Short* Division they are performed by the memory, and only the quotient is put down.

OBS. It is worth remarking, that in finding how often the divisor will go into the dividend or any of the rem^{rs}, we cannot always get a correct result by merely dividing the first figure of the divisor into the first, or first two figures of the dividend. Thus, in the first division of the example given above, though 4 in 17 goes 4 times, yet 4587 in 17465 will not go 4 times. Nothing but practice will enable a pupil to hit upon the correct number at once.

Exs. 20.

Find the required quotients in the following Examples.

1. 35689742 ÷ 23.
2. 1468920357 ÷ 29.
3. 3689740125 ÷ 37.
4. 41562389075 ÷ 39.
5. 3875401267 ÷ 41.
6. 7649801325 ÷ 43.
7. 2607458913 ÷ 47.
8. 146892750438 ÷ 53.
9. 689432167 ÷ 153.
10. 4987531681 ÷ 257.
11. 1728956043 ÷ 345.
12. 8653219547 ÷ 436.
13. 9875426817 ÷ 563.
14. 14576148903 ÷ 684.
15. 3875421986 ÷ 796.
16. 45862175647 ÷ 891.
17. 16894321678 ÷ 1531.
18. 3459806546 ÷ 4270.
19. 98423614578 ÷ 1094.
20. 57302648192 ÷ 4326.
21. 451728954320 ÷ 5783.
22. 52198736929 ÷ 68754.
23. 10943268759 ÷ 43281.
24. 94544671038 ÷ 96329.

28. There is one kind of divisors with which, though greater than 12, we can yet divide in one line, as 20, 30, 40, 500, 800, &c.; that is, the numbers 1 to 12 followed by one or more ciphers.

For instance, to divide 37458 by 20. Performing the operation by Long Division, we have the work as (F): or

if we cut off the cipher in the 20, and the last figure in the dividend, namely 8, the work would stand as in (G).

(F)	(G)
20) 37458 (1872	2,0) 3745,8 (1872
20	2
174	17
160	16
145	14
140	14
58	5
40	4
18	1

And if the 8 which was cut off be now brought down to the last rem^r 1, we shall have the same quotient and rem^r as in the former operation; hence this second operation is correct, and we may thus put the work of (G) in the form of Short Division,

2,0) 3745,8
1872 „ 18 rem^r

where, in performing this division, we use the last rem^r 1 as though it were 10, and adding the 8 which was cut off, make the whole rem^r 18. And this appears to be reasonable; for since we have used the divisor 20 as though it were 2, therefore a rem^r 1 is to be counted as 10.

In like manner, if I divided by 200, or 300, I should cut off *two* figures from both divisor and dividend, and divide by 2 or 3; if by 2000, or 3000, I should cut off *three* figures, and divide by 2 or 3. If by 1000, I have only to cut off three figures from the dividend for a remainder, and keep the figures not cut off as a quotient.

2,00) 387,59
193 „ 159 rem^r

1,000) 986,421
986 „ 421 rem^r

7,000) 5435,189
776 „ 3189 rem^r

Exs. 21. Divide

I. 38749102 by 10, 100, 1000.

II. 258749157 by 20, 30, 40, 500, 6000.

III. 78542963 by 900, 8000, 500,000.

It was shown in (22) that the product of the numbers representing the length and breadth of an oblong or square, in feet or inches, gave us the number of square feet or inches in the surface. So, also, if the number of square feet or inches in a surface, and the length of it, be known, the breadth will be found by dividing the surface by the length. Of course, if the breadth be given, the length can be found by dividing the surface by the breadth.

Hence we know the following facts.

$$\text{Length} \times \text{breadth} = \text{oblong surface.}$$
$$\text{Surface} \div \text{length} = \text{breadth.}$$
$$\text{Surface} \div \text{breadth} = \text{length.}$$

Exs. 22.

1. If there are 120 square inches in a surface, and it be 12 inches long, how broad is it?
2. There are 3625 square inches in the surface of an oblong slab, and it is 25 inches broad, how long is it?
3. A box measures 75 inches round, and its sides contain 2812 square inches, what is its depth?
4. The same box is 20 inches long, how many square inches in the top and bottom?
5. A board is 15 inches broad, how long must it be that there may be 450 square inches in *both* its surfaces?
6. An oblong plantation contains 175 trees planted regularly one foot apart; if it contain 25 in length, how many in breadth? If 35 in length, how many in breadth?
7. How many feet round will the above two oblongs be?

Also, since

$$\text{Divisor} \times \text{quotient} + \text{remainder} = \text{dividend,}$$

therefore, when any three of these quantities are known, the fourth one can be found.

Exs. of this kind will be found below.

Exs. 23.

MISCELLANEOUS QUESTIONS INVOLVING THE SIMPLE RULES.

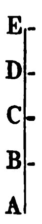
1. What number subtracted from 35890101 will leave 67842?
2. Divide 175 + 368 + 459 by 3685 − 3174.
3. The product of two numbers is 387659; one of them is 6432; what is the other?
4. How much does 384501 amount to, when repeated 999 times?
5. There are two numbers, the greater of which is three times 3728, and the less is twice 1479, what is the difference?
6. The divisor is 35, the quotient 38975, and remainder 17, what is the dividend?
7. Twelve hundred and ten workmen receive among them in 3 months £18150; how much is that for each workman per month?
8. How many fifties are there in five hundred and ten millions?
9. The sum of two numbers is 38976, and one of them is 3459, what is the other?
10. The sum of two numbers is 45873, and the greater of them is 31267, what is their product?
11. What is the difference between the 11th and 12th parts of 42768?
12. If light travels at the rate of 192,000 miles per second; how far must the sun be from us, if his light is 490 seconds reaching us?
13. In a crew of 847 men, each man receives £3 a month; how much is paid to the whole crew in 12 months?
14. If 7848 marbles are divided equally among 72 boys, how many will each have?
15. A field in the form of a double square is 75 yards broad, how long is it, and how many square yards does it contain?
16. A wall is 120 bricks long, 17 high, and 3 thick, how many bricks does it contain?
17. In 27 bales of cloth, each containing 15 pieces, and each piece 15 yards, how many yards?
18. Shew that the product of 3846 and 705 is equal to the quotient of 51517170, when divided by 19.
19. Write the above fact in figures and signs.
20. The sum of 368979 and 335342 is equal to the sum of the products of 85 and 709, and of 11501 and 56. Give the value of these quantities, and write the expression in figures and signs.
21. A book contains 215 pages, 55 lines in a page, and 45 letters in a line; how many lines and letters in the book?
22. The 12th part of a number is 7563; what is the number itself?
23. A certain number when divided by 75 is 8907; what is the number?

24. The number 6742 when multiplied by a second number becomes 5771152, what is the multiplier?
25. The dividend is 73992, the remainder 242, and the quotient 202, what is the divisor?
26. Express in signs these words, "the difference between the quotients of 37200 divided by 496, and of 45692 divided by 23 is 117."
27. If the quotient, divisor, and remainder be given, how do you find the dividend? Ex. Find the dividend, when the quotient is 345, the divisor 178, and remainder 27.
28. If the dividend be 4487234752, and the quotient 64064, what is the divisor?
29. By how much is the sum of 13459 and 756 greater than the sum of 1001 and 897?
30. How much greater is the product of 1894 and 325, than their sum?
31. From one million I take away 1000, and divide the remainder into 25 equal parts; how many in each part?
32. Fifty persons contribute 29 articles each; forty others give 27 each; and ten others 17 each; how many in all?
33. Express the result of $36 + 4 \times 9 + 17$.
34. From three thousand and one take 299; multiply the remainder by 75 and 25 successively; what is the result?
35. Write out in words, $184 + 36 - 201 + 99 = 245 - 83 + 49 - 93$. What is the value of each of these equal quantities?
36. A book contains 275 pages of large type, with 35 lines in a page, and 45 letters in a line; and 97 pages of small type, with 55 lines in a page, and 67 letters in a line; how many lines and letters are contained in the book?
37. If the divisor be 375, the quotient 4655, and the dividend 1756976, find the remainder, without dividing.
38. The Creation took place 4004 B. C.; how many periods of 177 years, from that time to the end of the 60th year of the 17th century.
39. In a regiment consisting of 875 men, there are 5 officers to every 120 privates; how many officers in all? and how many privates to one officer?
40. Explain the method of dividing by a number which is formed by the multiplication of two numbers, each less than 12. Shew how to form the complete remainder.

OBSERVATIONS INTRODUCTORY TO

REDUCTION AND COMPOUND RULES.

29. We have so far been dealing only with the numbers described in Arts. (1) to (9); that is, with *whole* numbers; and the smallest number mentioned has been 1, or unity. We now come to numbers which are either less than 1, or are between any two adjoining numbers, as between 2 and 3, 7 and 8, &c. Of this sort are the numbers which express the value of the familiar quantities, seven pence halfpenny, two yards and three quarters, &c. Such numbers are called *fractional*. But as a complete explanation of fractions would be generally found too difficult for pupils who have only just mastered the simple rules, we shall therefore give the names and meaning only of those fractional quantities which are most commonly used in Reduction and the Compound Rules. They are *one quarter; two quarters*, or *one half;* and *three quarters.* These common divisions may be thus explained.



Take a line ABCDE, one inch long, and divide it into two equal parts at C. Next, divide AC into two equal parts at B, and CE into two equal parts at D. The whole line AE will now have been divided into 4 equal parts, AB, BC, CD, DE, which are commonly called *quarters.* Also, if from A to B is *one* quarter, from A to C will be *two* quarters, from A to D will be *three* quarters, and from A to E will be *four* quarters, which make up the whole AE. The line AC which we see is *two* quarters, is generally called one-half.

Hence the three principal divisions to be remembered are

one quarter,	two quarters, or, one-half,	three quarters,

and they are thus written in figures:

$\frac{1}{4}$, $\frac{2}{4}$, or $\frac{1}{2}$, $\frac{3}{4}$.

Thus $7\frac{1}{4}$ inches is read seven and a quarter inches, or seven inches and a quarter.

If, instead of dividing one inch, I had divided one penny, or any other single article, into 4 equal parts, I should have written the fractional parts in precisely the same manner. Also, a penny has been divided into 4 smaller coins, called farthings: and since these 4 coins are *quarters* of 1 penny, therefore the word *qrs.*, which is short for quarters, is often used to represent farthings.

Hence, since farthings are quarters of 1 penny, we shall have

1 far. or *one* qr. of a penny = $\frac{1}{4}$ of a penny,
2 far. or *two* qrs. „ = $\frac{2}{4}$, or $\frac{1}{2}$ of a penny,
3 far. or *three* qrs. „ = $\frac{3}{4}$ of a penny.

Instead of writing the words *of a penny*, as I have done, we write the letter *d*;* thus $\frac{1}{4}$d. means one-fourth of a penny: and 7d. means 7 pence; so $7\frac{3}{4}$d. means 7 pence and 3 quarters of a penny, or 7 pence three farthings.

30. A quantity which consists of several kinds or denominations is called a *Compound* quantity. Thus, 25 pounds, 14 shillings, and 3 pence, is called a compound quantity; and it is written thus; £25 14s. 3d.; a space being placed between the pounds, shillings, and pence, that they may be kept distinct.

* This letter *d* is the first letter of the Latin word *denarius*, the Roman *penny*, as it is sometimes called; but the coin was in reality equal to about $7\frac{1}{2}$ pence of our money.

REDUCTION.

PART I.

31. Reduction is the changing of quantities which are expressed by numbers, from one or more denominations, to one or more others, so that the actual values of the quantities shall remain unaltered.

Before the method of performing these changes can be understood, it is necessary for the pupil to learn what are called the TABLES of Money, Weights, and Measures. We give one of the simplest for the sake of working examples with it.

2 farthings	=	1 halfpenny,	or ½d.
2 halfpence or 4 farthings	=	1 penny,	or 1d.
12 pence	=	1 shilling,	or 1s.
20 shillings	=	1 pound sterling,	or £1.

32. As an Example of the process of reduction, let it be required to reduce £50 to shillings.

Now we know that £1 contains 20 shillings; therefore, for every pound in the £50 we must have 20 shillings; that is, we must have in all 20 times as many shillings as we have pounds, or, 20 times 50 pounds. If, then, we multiply the £50 by 20, we have as the product the number of shillings in £50, namely 1000. The work will be as above.

$$\begin{array}{r l} \pounds & \\ 50 & \\ 20 & \\ \hline 1000 & \text{shillings.} \\ \hline\hline \end{array}$$

Again, if it be required to reduce the £50 to pence;

then, since there are 12 pence in 1 shilling, there will be 12 times as many pence as there are shillings. We have already seen that there are 1000 shillings in £50. If, therefore, we multiply the 1000 shillings by 12, we shall have a product of 12000, which is the number of pence in 1000 shillings, or in £50.

$$\begin{array}{r} \pounds \\ 50 \\ 20 \\ \hline 1000 \text{ sh.} \\ 12 \\ \hline 12000\text{d.} \\ \hline\hline \end{array}$$

And if we had further to reduce the £50 to farthings; then, since there are 4 farthings in 1 penny, if we multiply this 12000 pence by 4, we shall have 4 times as many farthings as pence, or 48000 farthings in the £50.

The whole work of the above example is represented in the margin; and it teaches how to reduce quantities of a *higher* name, as pounds, to quantities of a similar kind, but a *lower* name, as shillings, pence, and farthings.

$$\begin{array}{r} \pounds \\ 50 \\ 20 \\ \hline 1000 \text{ sh.} \\ 12 \\ \hline 12000\text{d.} \\ 4 \\ \hline 48000 \text{ far.} \\ \hline\hline \end{array}$$

33. In like manner, if we had to reduce any other quantity, as tons, to any lower denominations, as hundred-weights, quarters, pounds, &c., we should multiply by the numbers given in the TABLES which join the successive denominations. Thus, in reducing tons to hundred weights, the multiplier is 20, because there are 20 cwts. in 1 ton; from hundred weights to quarters, the multiplier is 4, since there are 4 quarters in 1 cwt.; from quarters to lbs. it is 28, since there are 28 lbs. in 1 quarter; and so on.

Exs. 24.

1. Reduce £75 to pence.
2. Reduce £135 to farthings.
3. Reduce 520 guineas to pence.
4. Change 1075 moidores to pence.

5. How many farthings in a £10 note?
6. Reduce 15 cwt. to ounces.
7. Reduce 13 tons to drams.
8. How many drams (Apothecary) in 15 lbs.?
9. Reduce 17 lbs. Troy to pennyweights and grains.
10. Reduce 750 yards to nails.
11. How many seconds in five weeks?
12. Find the number of sheets of paper in 755 reams.

34. Sometimes the quantity given to be reduced is a compound quantity, as £25 13s. 6¾d., to be reduced to farthings.

Here we proceed to multiply by 20, 12, and 4, as in the Example just worked; but when we reduce to shillings, we add the 13s. into the line of shillings; so, in reducing to pence, we add the 6d. to the line of pence; and lastly, we add the 3 farthings into the line of farthings. The whole work will be most easily understood as in (E), but we commonly write it as in (F).

(E)

£	s.	d.	
25	13	6¾	
20			
513 sh.			= £25 13s.
12			
6162d.			= £25 13s. 6d.
4			
24651 f.			= £25 13s. 6¾d.

(F)

£	s.	d.
25	13	6¾
20		
513 sh.		
12		
6162d.		
4		
24651 f.		

And in like manner for any other compound quantity. Hence, for reducing quantities to a lower denomination, we have this

Rule. Multiply the highest denomination by the number given in the tables as connecting it with the next lower; and if in the quantity to be reduced, there be any units of this lower denomination, add them to the product; repeat this step for every succeeding denomination, till the quantity is reduced to the required name.

Exs. 25.

1. Reduce £75 15s. 6d. to pence.
2. Reduce 19 guineas 18s. $7\frac{1}{4}$d. to farthings.
3. Convert 157 moidores 13s. 4d. to pence.
4. Change 3255 crowns 2s. $6\frac{3}{4}$d. into pence and farthings.
5. How many drams in 3 tons 0 cwt. 2 qrs. 17 lbs. 14 oz. 5 drs.?
6. How many grains in 17 lbs. 3 dwts. 15 grains?
7. Express 75 gals. 3 qts. 1 pt. of wine in half pints.
8. Change 175 lbs. 7 oz. 3 drs. 2 scr. into scruples.
9. In 117 sacks 11 pks. 1 gal. 3 qts., how many quarts?
10. Reduce 377 Eng. ells 4 qrs. 3 nls. to nails.
11. How many minutes in 365 days 5 hrs. 48 min.?
12. Find the number of poles in 817 mls. 5 fur. 25 poles.

35. Ex. III. Reduce £315 15s. 4d. to crowns and twopences. Here, the denominations are not those usually found in Tables of Money; but since we may observe that there are 4 crowns in £1, and 30 twopences in one crown, we therefore multiply, first by 4, and then by 30.

Also, since in 15s. there are three crowns, I add this 15s. into my first product, viz. of crowns, not as 15, but as 3: and in the next product, viz. of twopences, I add the 4d. not as 4, but as 2. The work is

£	s.	d.
315	15	4
4		
1263 crowns		
30		
37892 twopences.		

Exs. 26.

1. Reduce £345 15s. to crowns.
2. Change £5001 17s. 6d. to half-crowns.
3. Convert 1755 guins. 19s. 6d. to sixpences.
4. How many groats in £371 18s. 8d.?
5. What number of twopences shall I receive for 175 moid. 23s. 10d.?
6. How many parcels of 4 oz. in 17 cwt. 2 qrs. 17 lbs.?
7. Bring 3 weeks 6 days 19 hrs. into spaces of 10 minutes.
8. How many parcels of 6 sheets are contained in 175 reams 15 quires of paper?

All these Examples have required only multiplication, because in every case we had to change our larger coins into smaller, and therefore we required *more* in number. Hence we multiplied.

36. Now let it be required to perform an operation of reduction just the reverse of that employed in the above Examples; for instance,

Ex. IV. To reduce 1225 farthings to pounds.

Since 4 farthings = 1 penny, we shall have but 1 penny for every 4 of the 1225 farthings; therefore the whole number of pence therein will be found by seeing how often 4 is contained in 1225, that is, by dividing it by 4; and we find that there are 306 pence, and a remainder 1, which is of course 1 farthing, or $\frac{1}{4}$d., since it is one of the 1225 farthings.

Again, if it be required to bring the 1225 farthings, or 306 pence, to shillings; since there will be only one shilling for every 12 of the pence, we shall obtain the number of shillings in 306 pence by dividing the 306 by 12; this gives 25 shillings, and a remainder 6, which is of course 6 pence. If we wish to reduce to pounds, we have in like manner only 1 pound for every 20 shillings, and must therefore divide the 25 by 20: this gives a quotient £1, and 5 over, which is 5 shillings. The complete quotient with all the remainders is £1 5s. 6$\frac{3}{4}$d. The operation stands thus.

$$
\begin{array}{l}
\text{farthings} \\
4)\ \underline{1225} \\
\ \ 12)\ \underline{306}\ \ \tfrac{1}{4}\text{d.} \\
\ \ \ \ 2{,}0)\ \underline{2{,}5}\ \ 6\text{d.} \\
\ \ \ \ \ \ \ \ \underline{\underline{\pounds 1\ \ 5\text{s.}\ \ 6\tfrac{1}{4}\text{d.}}}
\end{array}
$$

37. If the quantity given to be reduced were of any

other name,—as ounces avoirdupois, to be brought to cwts.,—we must first divide by 16, to bring ounces to lbs. since 16 oz. = 1 lb.; then by 28, to bring lbs. to quarters, since 28 lbs. = 1 quarter; and lastly by 4, to bring quarters into cwts., since 4 qrs. = 1 cwt.

Hence, if we have to reduce a quantity from a lower denomination to a higher, we have this

RULE. Divide the proposed quantity by the number which is given in the tables, as connecting it with the next higher name; and place the remainder, if any, on a line with the quotient, and with its name attached to it. Perform similar operations of division, till the quantity is reduced to the required name. When the last quotient is obtained, bring down in a line with it all the remainders in their proper order, beginning with the highest in value.

Exs. 27.

1. In 34758 pence, how many pounds?
2. In 75389 farthings, how many guineas?
3. Change 137456 pence for moidores.
4. How many lbs. Troy in 756843 grains?
5. Reduce 2374598 seconds to days and weeks.
6. In 41063897 drams, how many tons?
7. What number of quarters is contained in 7410683 pints?
8. How many days in 89765321 seconds?
9. Reduce 7589432 pints of wine to hogsheads.
10. Reduce 13897564 minutes to years.
11. Find the number of lbs. Apoth. in 38596 grains.
12. How many barrels of ale in 47891 half pints.

38. Sometimes, as in Ex. V., the denominations given or required may not be those generally found in the tables: We have then only to find the divisors which connect every two successive denominations, and be careful to observe the nature of the remainders.

Ex. V. Reduce 38975 groats to crowns and pounds.

Here, since 3 groats = 1 shilling, I must first divide by 3, and the remainder will be groats. Also, since 5 shillings = 1 crown, I must next divide by 5, and the remainder will be shillings. And, since 4 crowns = £1, I must next divide by 4, and the remainder will be crowns.

The work is as follows.

```
        groats
     3) 38975
     5) 12991  2 gr.
      4) 2598  1 sh.
          £649 2 cr.,   or £649 2 cr. 1 sh. 2 groats.
```

We have two answers, £649 11s. 8d., or 2598 cr. 1s. 8d.

Exs. 28.

1. Change 34275 groats into moidores.
2. How many crowns in 145893 farthings?
3. How many sixpences and half-crowns in 58976 farthings?
4. Change 34589 half-crowns into pounds.
5. Convert 148235 twopences into moidores.
6. Change 348796 groats into seven-shilling pieces.
7. Change 38976 drams (Avoirdupois) into portions of 7 lbs. each?
8. How many spaces of 3 hours are contained in 49539600 seconds?
9. In 458976 parcels of 4 sheets of paper, how many reams?
10. How many measures of 3 hogsheads are contained in 48976 pints of wine?

NOTE. If a pupil, in attempting an Example in reduction, be in doubt whether to use the first or the second Rule, that is, whether to multiply or divide, he must remember that he has only to ask himself one question: Will the answer which I have to find be a number *greater* or *less* than the number which I have to reduce? If it is to be *greater*, I must of course multiply; if *less*, I must divide. Thus, to bring £10 to farthings, I must of course *multiply*, because there will plainly be more farthings than *ten* in £10; and if I have to reduce 1250 pence to pounds, I

must *divide*, because there are evidently fewer pounds than 1250, in 1250 pence.

39. In some of the Examples under weights and measures we shall find it necessary to multiply and divide by such quantities as $5\frac{1}{2}$, and $30\frac{1}{4}$.

I will give an Example of each of such cases.

Ex. VI. Reduce 3 fur. 17 po. 2 yds. 1 ft. to feet.

$$
\begin{array}{rl}
 & 3\text{ f. } 17\text{ p. } 2\text{ y. } 1\text{ ft.} \\
 & \underline{40} \\
 & 137\text{ po.} \\
 & \underline{5\frac{1}{2}} \\
 & 687 \\
\frac{1}{2}\text{ of } 137 = & \underline{68\frac{1}{2}} \\
 & 755\frac{1}{2}\text{ yds.} \\
 & \underline{3} \\
 & \underline{\underline{2267\frac{1}{2}}}\text{ feet.}
\end{array}
$$

The first multiplication by 40 is quite plain. Now to multiply a quantity by $5\frac{1}{2}$ is to repeat it 5 times and half a time; therefore $5\frac{1}{2}$ times 137 will be found by multiplying the 137 by 5, and then adding to this product one half of 137; thus working, and adding in the 2 yds., I obtain $755\frac{1}{2}$ yds. as the whole product. In bringing these yards to feet by multiplying by 3, I remember that the $\frac{1}{2}$ yd. is 1 foot and a half; so adding it in, as well as the 1 foot in the top line, I obtain $2267\frac{1}{2}$ as the number of feet in 3 fur. 17 po. 2 y. 1 ft.

Had the multiplier been $5\frac{1}{4}$, instead of $5\frac{1}{2}$, then instead of taking one-*half* of the 137, I should have taken one *quarter*. The following Example will illustrate this.

Ex. VII. Reduce 3 ro. 23 po. 14 sq. yds. to square feet.

$$
\begin{array}{rl}
 & 3\text{ ro. } 23\text{ po. } 14\text{ sq. yds.} \\
 & \underline{40} \\
 & 143\text{ po.} \\
 & \underline{30\frac{1}{4}} \\
 & 4304 \\
\frac{1}{4}\text{ of } 143 = & \underline{35\frac{3}{4}} \\
 & 4339\frac{3}{4}\text{ sq. yds.} \\
 & \underline{9} \\
 & \underline{\underline{39057\frac{3}{4}}}\text{ sq. ft.}
\end{array}
$$

Multiplying by 4, and $30\frac{1}{4}$, I obtain $4339\frac{3}{4}$ square yards, and since the $\frac{3}{4}$ sq. yd. is equal to $6\frac{3}{4}$ square feet, I have added it in as $6\frac{3}{4}$, when multiplying by 9 to reduce to square feet.

40. Now let it be required to *divide* by these same numbers $5\frac{1}{2}$ and $30\frac{1}{4}$. The following Examples will require such divisions.

Ex. VIII. Reduce 384976 feet to furlongs.

```
        feet
  3) 384976
5½) 128325   1 ft.
 2        2
11) 256650
4,0) 2333,1  9 hf. yds.
     583 fur. 11 po. 4½ yds. 1ft.
```

Here, proceeding to reduce to yards, poles, and furlongs successively, I divide by 3, $5\frac{1}{2}$, and 40. The first division is simple. In the second division I have to see how often $5\frac{1}{2}$ yds. are contained in 128325 yds. Now I cannot divide by $5\frac{1}{2}$, as it stands; but if I bring both divisor and dividend into half yards, viz. 11 and 256650, it will be precisely the same, whether I see how often $5\frac{1}{2}$ yds. are contained in 128325 yds., or 11 *half* yds. in 256650 *half* yards.

Performing the division, I have as quotient 23331 poles; and because the dividend 266650, was half yards, therefore the remainder 9 was 9 half yards, or $4\frac{1}{2}$ yds. Dividing as usual by the 40, to bring poles into furlongs, I have the complete answer 583 fur. 11 po. $4\frac{1}{2}$ yds. 1 ft.

41. The following Example will show how to divide by $30\frac{1}{4}$, and will need no explanation.

Ex. IX. Reduce 785447 sq. feet to acres.

```
                 sq. feet
yds.          9) 785447
30¼               87271  8 sq. ft.
 4                    4
121 qrs  { 11) 349084 qrs.
         { 11)  31734  10 }
          4,0)  288,4  10 } 120 quarters, or 30 yds.*
             4) 72    4 poles
                18 ac. 0 ro. 4 po. 30 sq. yds. 8 sq. ft.
```

* The reader who is acquainted with fractions will perceive that this method is

Exs. 29.

1. How many square inches are contained in 3 ro. 35 po. 25 yds. 8 ft.?
2. Reduce 15 m. 7 fur. 8 po. 3 yds. to yards.
3. Convert 185 degrees of $69\frac{1}{2}$ miles into yards.
4. In 1815 coins, each worth $5\frac{1}{4}$ guineas, how many pence?
5. Convert 37584 dollars, each worth $4\frac{1}{2}$ shillings, into halfpence.
6. Convert 60000 barley corns into fathoms.
7. In 37589 inches, how many fathoms?
8. Reduce 13859 yds. to miles?
9. How many leagues in 478321 feet?
10. How many acres in 349876 square poles?
11. Reduce 6897543 square inches to roods.
12. In 4596328 perches, how many square miles?

42. But there is another class of Examples in Reduction which require both the above processes of Multiplication and Division to be used in the same question. As a simple Example of this kind, let us take

Ex. X. To reduce 1000 guineas to pounds.

Now this question really is; "How many times is £1 contained in 1000 guineas?" To answer this, I must divide 1000 guineas by £1; but I cannot do this, till I bring both divisor and dividend to the same name. The highest coin of which they both consist is shillings; I therefore reduce them both to shillings and then divide.

The operation is most clearly shown thus,—

$$\begin{array}{rr} & \text{guineas.} \\ & 1000 \\ & 21 \\ \hline \text{£}1 & 1000 \\ 20 & 2000 \\ \hline 2,0\text{sh.}) & 2100,0\text{sh.} \\ \hline & 1050 \text{ pounds.} \end{array}$$

---

identical with that pursued in division of fractions.

$$\text{Thus, } 87271 \text{ yds.} \div 30\tfrac{1}{4} = \frac{87271}{\frac{121}{4}} \text{ po.} = \frac{87271 \times 4}{121} \text{ po.}$$

$$= \frac{349084}{121} \text{ po.} = 2884\tfrac{120}{121} \text{ po.}$$

43. In this Example we have had to reduce the given quantity, guineas, and the required quantity, pounds, only one step,—namely, to shillings. But sometimes the numbers will require to be reduced more than one step, as for instance, if it be required to change this 1000 guineas into half-crowns.

```
                 guineas
                   1000
                     21
                   ----
                   1000
 s.  d.            2000
 2   6            -----
                  21000 sh.
 12                  12
 ---              -------
 3,0 pence)       25200,0 pence.
                  8400 hf. crowns.
```

I have here to see how often a half-crown will go in 1000 guineas. Since 1000 guineas and 1 half-crown both exactly consist of pence, I bring both to pence, and they become 252000d. and 30d. Dividing the greater by the less, I have as quotient 8400, which is the required number of half-crowns.

```
            guineas.
              1000
 s.  d.         21
 2   6       21000 sh.
 2               2
 -           -----
 5 sixp.)    42000 sixp.
              8400 hf. crowns.
```

Again, since 2s. 6d. and 1 guinea both consist of *sixpences*, I might have reduced them both to sixpences, instead of pence, and the work would then have been shorter.

For such Examples, we may therefore lay down the following

RULE. Find the greatest denomination or kind of which both quantities exactly consist; reduce them both to that denomination, and divide the greater by the less.

44. I give one more Example of this kind, on account of the difficulty that pupils sometimes have in applying the above Rule.

Ex. XI. In £453 16s. 8d., how many pieces of coin, each 3s. $4\frac{1}{2}$d.?

Here the coin or denomination of which both the quantities consist is half-pence. I therefore reduce both to half-pence, and, as before, divide the larger number by the smaller: The remainder 30 must of course be half-pence; so that the required number of coins is 2689; and 15½d. remain.

s.	d.		£	s.	d.
3	4½		453	16	8
12			20		
40d.			9076 sh.		
2			12		
81 hf. pence			108920d.		
			2		
			9) 217840 hf. pence.		
			9) 24204	4	30 hf. pence.
			2689	3	

Exs. 30.

1. In £50 15s. 6d., how many coins each 4s. 6d.?
2. Reduce 175 guineas to pieces each worth 2s. 7½d.
3. Find how many coins of 6s. 8d. can be obtained out of £217 19s. 8d.
4. I exchange 375 pieces of 7s. 6d. each for coins worth 22s. 9d. each, how many shall I obtain?
5. Divide 1875 yds. 3 qrs. into pieces, each 3¼ nails?
6. How many portions of 1¼ oz. can be obtained from 1 cwt. 1 qr. 6lbs?
7. What number of intervals of 2½ feet are there in 1 mile 800 yds.?
8. Find how many intervals of 19¼ seconds there are in a week.
9. How many distances each 1⅜ furlongs in a degree?
10. Convert 375 Eng. ells into portions of 4⅛ inches.
11. How many subscribers of 27s. 6d. each will be required to raise £1925 10s.?

45. We have now given Examples of all the *principles* which we need learn in order to work any question in Reduction; but there are such various forms in which Reduction can occur, that it will be advisable to work some additional questions which will illustrate the difficulties. It is better, however, to defer these questions till after the Compound Rules are mastered, because in many of them a knowledge of these rules is of use; also, the pupil will, by his increased experience, be more able to contend with difficult examples.

COMPOUND RULES.

ADDITION OF COMPOUND QUANTITIES, OR

COMPOUND ADDITION.

46. In adding Compound Quantities, we must remember as in Simple Addition, that like quantities must be added together, as farthings to farthings, pence to pence, and so on.

Hence, if it be required to find the sum of any compound quantities, as £73 2s. 9¼d.; £25 8s. 4¾d.; £68 3s. 11¼d.; £76 17s. 7d.; £5 14s. 5¾d.; we must place the quantities under one another so that the farthings may be in a row, as also the pence, shillings, and pounds: and the sum will stand thus:

£	s.	d.
73	2	9¼
25	8	4¾
68	3	11¼
76	17	7
5	14	5¾
249	7	2¼

Beginning at the right-hand column, and adding the farthings, we find their sum to be 9: this, by reduction to pence, gives 2 pence and 1 farthing over: put down the 1 farthing (thus, ¼) and carry the 2 pence to the next column, which consists of pence. Adding it, as in Simple Addition, we find its sum to be 38 pence, which by reduction gives 3 shillings, and 2 pence. Put down the 2 pence, and carry the 3 shillings to the next row which consists of shillings. Adding again, we find the row

of shillings to amount to 47 shillings, or, by reduction, to 2 pounds, 7 shillings: put down the 7s. and carry the £2 to the next column, which consists of pounds. The sum of this last column we find to be £249; place this under the pounds. The complete answer is therefore, 249 pounds seven shillings and twopence farthing, or £249 7s. 2¼d.

47. The method would be precisely the same, if the quantities to be added were expressed in any other table, weight, or measure; only that in carrying from any column to the next, we should have to use the reductions proper to the denominations we were adding: and the method of performing these reductions has been explained under the head of REDUCTION.

Hence, for adding together compound quantities, we have this

RULE. Arrange the quantities under one another, so that all those of the same kind may be in upright rows. Begin at the right-hand, and find the sum of the first row, as in Simple Addition: see how many units of the next higher name are contained in the sum, put down the remainder, if any, and carry those units to the next row; proceed in like manner with each column to the end, and the sum of the last column to the left write down in full.

Exs. 31. Find the value of

	£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
1.	16	8	9½ +	1106	19	7¼ +	915	10	0 +	3	8	7¾
	+5	6	8½ +	1415	5	2¾.						
2.	858	7	6¼ +	1984	19	11½ +	347	5	9¼ +	3426	17	10
	+549	9	9¾ +	1685	12	8¾.						
3.	345	7	9½ +	685	11	2 +	3899	14	11¾ +	7645	8	7¼
	+598	19	2½ +	1386	7	4.						
4.	3476	8	9¼ +	11063	10	9 +	9876	15	11½ +	10010	3	0¾
	+7856	16	7¼ +	15892	18	8¾.						

£ s. d. £ s. d. £ s. d. £ s. d.

5. 13487 12 2 + 9875 18 9¼ + 34267 5 6½ + 1899 9 9¾
+ 24682 11 7½ + 13897 15 9.

6. 89645 13 3 + 745627 19 7¼ + 99314 16 5¼ + 3875 0 11
+ 45683 16 8 + 1976 14 5.

7. 38976 5 4½ + 7043 17 9 + 689 16 4¾ + 14582 9 11¼
+ 38764 17 10 + 429 19 9½ + 1080 0 7¾.

8. 41987 16 7 + 112785 14 3½ + 98979 19 8¼ + 4568 16 11¾
+ 3897 17 3¼ + 31456 11 0 + 4289 7 3½.

9. 10689 15 6 + 39345 8 9¾ + 4786 13 7 + 98764 14 3¾
+ 111468 17 11½ + 3487 19 10¼ + 87562 17 7.

10. 67489 10 7¾ + 149876 19 9 + 348754 17 4½ + 689 11 0½
+ 87532 9 11 + 4986 1 7¾ + 15019 3 6¼ + 875 10 10

LONG MEASURE.

yds. ft. in. b.c. yds. ft. in. b.c. yds. ft. in. b.c. yds. ft. in. b.c.

11. 5 2 9 2 + 16 1 11 1 + 18 0 7 0 + 25 1 8 2 +
17 2 6 1 + 6 2 9 0.

12. 75 2 11 2 + 187 0 9 1 + 34 1 8 0 + 93 2 7 1 +
+ 106 1 4 2 + 85 2 6 1.

mls. yds. ft. in. mls. yds. ft. in. mls. yds. ft. in. mls. yds. ft. in.

13. 185 25 2 7 + 17 809 1 8 + 361 73 0 11 + 95 145 2 4
+ 84 698 2 3 + 603 45 2 9.

14. 684 116 2 6 + 1785 385 1 8 + 907 47 0 10 + 64 903 1 9
+ 7832 86 0 7 + 986 345 2 5.

fur. po. yds. ft. fur. po. yds. ft. fur. po. yds. ft. fur. po. yds. ft.

15. 37 35 3 2 + 18 17 1¼ 1 + 109 30 4½ 2 + 75 29 2 1
+ 1846 18 3½ 0 + 49 4 4¼ 0 + 168 31 3 2.

16. 185 15 4 1 + 76 25 3 2 + 349 35 2 0 + 608 17 0 1
+ 705 19 5 2 + 986 23 3 2 + 432 11 2 0.

TROY WEIGHT.

lbs. oz. dwts. grs. lbs. oz. dwts. grs. lbs. oz. dwts. grs. lbs. oz. dwts. grs.

17. 17 5 16 20 + 135 8 15 19 + 89 9 19 23 + 604 11 7 7
+ 73 7 10 18 + 496 10 13 19.

18. 345 11 17 21 + 1029 9 18 17 + 687 8 4 19 + 4321 5 7 14
+ 864 4 9 3 + 387 2 11 12.

lbs. oz. dwts. grs. lbs. oz. dwts. grs. lbs. oz. dwts. grs. lbs. oz. dwts. grs.

19. 879 4 17 15 + 1000 10 6 11 + 754 9 18 17 + 16 11 19 5 + 1910 7 13 9 + 875 8 5 14.

20. 715 11 17 19 + 684 9 18 5 + 1932 8 14 16 + 45 7 15 22 + 507 4 13 20 + 18 0 11 17.

AVOIRDUPOIS WEIGHT.

lbs. oz. drs. lbs. oz. drs. lbs. oz. drs. lbs. oz. drs. lbs. oz. drs.

21. 101 8 2 + 74 12 4 + 53 10 6 + 20 14 9 + 63 13 13 + 39 7 15 + 103 9 17.

22. 75 3 11 + 176 11 15 + 819 14 14 + 47 9 3 + 160 7 11 + 918 15 8 + 456 10 9.

cwt. qrs. lbs. oz. cwt. qrs. lbs. oz. cwt. qrs. lbs. oz. cwt. qrs. lbs. oz.

23. 15 3 27 15 + 175 2 25 13 + 1987 1 13 11 + 432 0 19 9 + 375 3 16 7 + 1689 2 18 8.

24. 350 3 16 9 + 75 2 15 11 + 917 1 25 15 + 6542 0 23 13 + 689 2 27 10 + 750 3 20 14 + 1897 2 20 12.

tons cwts. lbs. oz. tons cwts. lbs. oz. tons cwts. lbs. oz. tons cwts. lbs. oz.

25. 75 19 17 14 + 389 14 63 13 + 2648 15 97 12 + 750 9 100 11 + 684 8 15 7 + 3968 7 45 8 + 589 6 111 10 + 642 18 87 9

26. 387 17 45 9 + 49 18 75 15 + 604 3 89 14 + 138 15 3 11 + 1796 8 10 7 + 423 9 44 8 + 897 11 101 3 + 1145 19 19 2

APOTHECARIES' WEIGHT.

oz. drs. sc. grs. oz. drs. sc. gr. oz. drs. sc. gr. oz. dr. sc. gr.

27. 11 7 2 19 + 8 6 1 17 + 9 4 0 13 + 17 0 1 11 + 35 3 2 8 + 86 2 0 9 + 97 1 1 12.

28. 17 5 0 17 + 139 7 2 18 + 44 6 1 10 + 65 3 0 19 + 63 2 2 11 + 245 1 1 13 + 178 0 0 14.

lbs. oz. drs. sc. lbs. oz. drs. sc. lbs. oz. drs. sc. lbs. oz. drs. sc.

29. 117 11 6 2 + 73 10 7 1 + 1094 6 5 2 + 685 4 3 1 + 734 11 4 0 + 99 3 2 1 + 108 9 1 2.

30. 375 9 7 2 + 649 4 6 1 + 832 11 3 0 + 1048 8 7 1 + 756 7 5 2 + 89 6 4 2 + 635 5 2 0.

CLOTH MEASURE.

yds. qrs. nls. in. yds. qrs. nls. in. yds. qrs. nls. in. yrds. qrs. nls. in.

31. 15 3 2 2 + 75 2 3 $1\frac{1}{2}$ + 389 0 2 2 + 60 1 1 $1\frac{3}{4}$ 14 2 0 2 + 175 3 2 $0\frac{1}{4}$ + 87 1 3 $1\frac{1}{4}$.

yds. qrs. nls. in. yds. qrs. nls. in. yds. qrs. nls. in. yds. qrs. nls. in.

2. 375 3 2 1¼ + 408 2 3 2 + 96 1 0 1¼ + 235 0 1 2
87 2 3 0¾ + 591 3 2 1½ + 62 1 0 1.

WINE MEASURE.

hhds. gals. qts. pts. hhds. gals. qts. pts. hhds. gals. qts. pts. hhds. gals. qts. pts.

3. 60 45 2 0 + 3 36 3 1 + 14 7 2 0 + 28 60 1 1 +
179 49 3 0 + 14 37 2 1 + 5 25 1 0.

4. 45 15 3 0 + 236 45 2 1 + 87 57 1 0 + 95 16 3 0 +
215 7 2 1 + 64 19 3 0 + 93 27 2 1.

5. 15 50 2 0 + 175 53 1 1 + 64 49 3 1 + 815 7 2 1 +
76 18 1 0 + 193 25 3 1 + 42 37 2 1.

6. 75 35 2 1 + 137 43 1 1 + 94 17 3 0 + 216 9 1 1 +
189 51 2 0 + 76 19 3 0 + 39 24 0 1.

ALE AND BEER MEASURE.

hhds. gals. qts. pts. hhds. gals. qts. pts. hhds. gals. qts. pts. hhds. gals. qts. pts.

7. 236 45 2 1 + 45 15 3 0 + 87 50 1 0 + 115 7 2 0 +
95 16 3 1 + 64 19 3 1 + 93 27 2 0.

hhds. bar. kil. gals. hhds. bar. kil. gals. hhds. bar. kil. gals. hhds. bar. kil. gals.

8. 17 1 1 14 + 85 1¼ 0¾ 15 + 193 0¾ 1½ 9 + 207 1 1¼ 6 +
79 0½ 1¼ 17 + 101 0 0 8.

SQUARE MEASURE.

ac. ro. po. yds. ac. ro. po. yds. ac. ro. po. yds. ac. ro. po. yds.

9. 75 3 19 7 + 329 2 25 18 + 4869 1 16 6 + 459 0 35 5
+ 78 2 18 4 + 385 1 17 3 + 1217 0 18 2¾ + 876 3 19 3

0. 118 2 11 25 + 457 3 39 17 + 11892 0 7 4 + 8972 1 27 20
+ 3145 1 37 30 + 9864 2 19 10 + 4382 3 20 15½ + 4 3 2 2¼

sq. m. ac. yds. sq. m. ac. yds. sq. m. ac. yds. sq. m. ac. yds.

1. 1358 345 3758 + 964 27 897 + 875 495 1684 + 4809 605 329
+ 293 87 401 + 687 327 2348 + 5904 95 3729 + 1873 119 495

CUBIC OR SOLID MEASURE.

sol. yds. ft. in. sol. yds. ft. in. sol. yds. ft. in. sol. yds. ft. in.

12. 387 18 1000 + 9126 25 895 + 45 7 1643 + 821 19 27
+ 3437 11 5 + 89 6 1519 + 1000 26 372.

13. 4186 15 874 + 379 9 910 + 25 18 35 + 5804 6 1681
+ 1427 11 25 + 983 10 984 + 467 17 39.

PAPER.

	reams	qui.	sheets	reams	qui.	sheets	reams	qui.	sheets	reams	qui.	sheets
44.	75	19	23 +	468	7	17 +	1937	16	18 +	46	11	5 +
	289	9	19 +	310	8	22 +	898	13	21.			
45.	875	18	17 +	9832	5	18 +	459	19	20 +	1684	11	5 +
	359	10	19 +	1875	15	21 +	348	15	23.			

COMPOUND SUBTRACTION.

48. Here, as in Simple Subtraction, the quantities to be subtracted must be taken from others of the same kind; and, therefore, we arrange the two quantities as in Compound Addition, putting the less under the greater.

Ex. Let it be required to find the difference of £325 19s. $4\frac{3}{4}$d. and £253 7s. $6\frac{1}{2}$d.

Placing them as we have just directed, and beginning at the right-hand, we take $\frac{1}{2}$d. or 2 farthings from $\frac{3}{4}$d., or 3 farthings, and the difference 1 farthing, or $\frac{1}{4}$d., we put down

£	s.	d.
325	19	$4\frac{3}{4}$
253	7	$6\frac{1}{2}$
£72	11	$10\frac{1}{4}$

under the column of farthings. Proceeding to the pence, we cannot take the 6 pence in the lower line from the 4 pence in the upper; we must, therefore, *borrow*, as it is called, from the next higher name, which is in this case shillings; we take 1 shilling, or 12 pence, from the 19 shillings, and add it to the 4 pence in the top line, making 16 pence: we now subtract the 6d. from the 16d., and have the remainder 10d., which is to be placed under the column of pence. And when we have borrowed in any subtraction, we must for the reasons given in Simple Subtraction, carry one to the next row to the left, and then subtract, We

shall thus subtract 8s. from 19s., and have a remainder 11s. In the pounds, we find the difference of the two rows, precisely as in Simple Subtraction, to be 72. Hence the whole difference is £72 11s. 10¼d.

49. The same alterations must be made in performing the subtraction as are made in Compound Addition, when the compound quantities consist of any other kind than money. Thus, in subtracting the pence, in the above Ex., we borrowed 12 pence, because 12 pence = 1 shilling; so, if the Example had been Avoirdupois weight, and while subtracting in ounces, we were obliged to borrow, we should have borrowed 16, because 16 ounces make 1 pound, which is an unit of the next higher name. And similarly for any other weight or measure.

Hence, if it be required to find the difference of two compound quantities, we have this

Rule. Place the less number under the greater, so that quantities of the same name may be under one another. Begin at the right-hand, and take the lower number from the upper if possible; but if the lower number be greater than the upper, take one unit of the next higher name, reduce it to the denomination in which you are now subtracting, add it to the upper figure, and then subtract, placing the difference underneath. Carry 1 to the lower figure of the next name, and proceed in exactly the same manner to the last figure on the left-hand.

Exs. 32.

	£	s.	d.		£	s.	d.		£	s.	d.		£	s.	d.
1.	85	9	7½	–	75	16	9	5.	8972	18	6½	–	4389	10	8¾
2.	432	19	8	–	375	0	11¼	6.	18759	11	9	–	9867	18	7¼
3.	1827	5	6	–	1103	18	9¼	7.	38974	15	6	–	9368	16	11½
4.	4268	19	0	–	575	19	10½	8.	14589	11	9¼	–	7892	10	11¾

	£	s.	d.		£	s.	d.
9.	45860	3	6¼	–	8977	7	10½
10.	46932	17	0	–	7145	18	7¼
11.	897654	18	11	–	69596	18	11½
12.	1489765	14	8	–	13609	17	9¼

TROY WEIGHT.

	lbs.	oz.	dwts.	grs.		lbs.	oz.	dwts.	grs.
13.	175	1	16	13	–	89	10	13	20
14.	8345	6	17	9	–	689	4	18	21
15.	5689	4	13	11	–	3870	8	9	17
16.	14896	8	11	10	–	9738	10	16	23

AVOIRDUPOIS WEIGHT.

	tons	cwts.	qrs.	lbs.		tons	cwts.	qrs.	lbs.
17.	2345	11	2	17	–	879	18	3	26
	cwts.	qrs.	lbs.	oz.		cwts.	qrs.	lbs.	oz.
18.	7189	2	15	8	–	349	3	19	12
	cwts.	lbs.	oz.	drs.		cwts.	lbs.	oz.	drs.
19.	3459	101	11	8	–	783	99	14	15
	cwts.	qrs.	lbs.	oz.		cwts.	qrs.	lbs.	oz.
20.	3195	85	7	11	–	839	103	15	18

CLOTH MEASURE.

	yds.	qrs.	nls.	in.		yds.	qrs.	nls.	in.
21.	3894	2	1	1¼	–	986	3	3	2
	E. ells	qrs.	nls.	in.		E. ells	qrs.	nls.	in.
22.	175	3	2	1	–	68	4	3	2
	Fr. ells	qrs.	nls.	in.		Fr. ells	qrs.	nls.	in.
23.	346	1	1	0	–	89	5	3	2
	Fl. ells	qrs.	nls.	in.		Fl. ells	qrs.	nls.	in.
24.	4185	2	1	1	–	376	2	3	1½

WINE MEASURE.

	bar.	gals.	qts.	pts.		bar.	gals.	qts.	pts.
25.	1375	12	2	0	–	889	15	3	1
26.	2387	24	1	0	–	748	30	3	1
	hhds.	gals.	qts.	pts.		hhds.	gals.	qts.	pts.
27.	8976	8½	25	2	–	987	1	33	3
28.	2892	50	1	1	–	983	34	2	0

SQUARE MEASURE.

	sq. yds.	ft.	in.		sq. yds.	ft.	in.
29.	7181	3	45	–	923	8	104
30.	8432	7	125	–	2785	8	37

	ac.	ro.	sq. po.	sq. yds.		ac.	ro.	sq. po.	sq. yds.
31.	375	2	19	25	–	17	3	35	30
32.	876	3	15	18¼	–	437	3	31	20

TIME.

	dys.	hrs.	min.	sec.		dys.	hrs.	min.	sec.
33.	245	13	14	53	–	67	19	41	17
34.	327	17	13	34	–	248	6	50	45

	wks.	dys.	hrs.	min.		wks.	dys.	hrs.	min.
35.	315	4	17	39	–	76	5	22	35
36.	178	3	3	57	–	109	6	19	59

DRY MEASURE.

	wys.	qrs.	bush.	pks.		wys.	qrs.	bush.	pks.
37.	73	4	5	3	–	45	1	7	2
38.	179	0	4	2	–	138	4	6	3

	lasts	wys.	qrs.	bush.		lasts	wys.	qrs.	bush.
39.	325	1	4	3	–	89	1	4	7
40.	1827	0	3	2	–	809	1	3	6

COMPOUND MULTIPLICATION.

48. As Simple Multiplication was shewn to be merely a shorter method of performing Simple Addition, so when we have learnt how to add compound quantities of a similar kind, we shall have no difficulty in multiplying compound quantities by any multiplier whatever. And first let the multiplier be not greater than 12; and let it be required to multiply £358 4s. 7¾d. by 5. We may perform the required operation both by addition and multiplication, and explain the second method from the first.

£	s.	d.
358	4	$7\frac{3}{4}$
358	4	$7\frac{3}{4}$
358	4	$7\frac{3}{4}$
358	4	$7\frac{3}{4}$
358	4	$7\frac{3}{4}$
£1791	3	$2\frac{3}{4}$

£	s.	d.
358	4	$7\frac{3}{4}$
		5
£1791	3	$2\frac{3}{4}$

Beginning with the farthings in either of the two sums, we have the sum of the farthings, or five times $\frac{3}{4}$d. = 15 farthings, which = 3 pence, and 3 farthings over; put down the farthings, and carry the 3 pence: so, also, 5 times 7d. = 35d., and with 3d. carried = 38d., or 3s. 2d.: 5 times 4s. = 20s., and with 3s. carried = 23s., or £1 3s.: and by Simple Multiplication we have 5 times £358, with the £1 carried, = £1791.

49. What has been said in Compound Addition about carrying, when the given quantities consist of any other than pounds, shillings, and pence, applies in Compound Multiplication; for we have shewn the two processes of Addition and Multiplication to be the same.

Hence, for multiplying any compound quantity by a multiplier under 12, we have this

RULE. Place the multiplier under the lowest denomination in the multiplicand, and multiply that name by the multiplier. See, as in Compound Addition, how many units of the next higher name are contained in this product; put down the remainder, if any, and carry these units to the next name. Multiply in like manner each denomination of the multiplicand: and when the highest product has been found, write it down in full.

50. Now let the multiplier be between 12 and 144. And first, let it be a number which can be exactly split into two numbers, for instance 72. This is equal to 8 times 9. If,

therefore, I multiply by 8, the first product will be equal to 8 times the multiplicand. If, now, I multiply this product by 9, then the second product will be 9 times the former one, or 72 times the top line,

£	s.	d.	
358	4	7¾	
		8	
2865	17	2	= 8 times
		9	
£25792	14	6	= 72 "

and therefore be the required amount. But if the multiplier cannot exactly be split into two numbers each under 12, as for instance 76, we must choose the next number below, which *can* be so split; in this case we take 72; and since 76 = 72 + 4, we may multiply by 72, or 8 × 9 as before, and then take 4 times the top line; thus

£	s.	d.	
358	4	7¾	
		8	
2865	17	2	= 8 times
		9	
25792	14	6	= 72 "
1432	18	7	= 4 "
£27225	13	1	= 76 "

obtaining 72 times, and 4 times the given quantity; the two products added together will give 76 times the multiplicand, as in the Example here given.

Exs. 33. Form the following products.

	£	s.	d.	
I.	7	18	10½	by 5, 6, 7, 8.
II.	13	9	7¾	by 7, 8, 9, 10.
III.	456	18	7¼	by 9, 10, 11, 12.

Exs. 34.

	£	s.	d.			£	s.	d.	
1.	138	19	4	× 35.	5.	498	17	10½	× 84.
2.	725	6	8½	× 24.	6.	2784	13	1¾	× 96.
3.	1829	2	11	× 40.	7.	3345	9	10½	× 108.
4.	1785	15	9¼	× 56.	8.	7864	17	8	× 120.

51. Next, let the multiplier be any whole number greater than 144, as 256.

Since $256 = 200 + 50 + 6$, if therefore we multiply the given quantity by 200, and by 50, and by 6, and add together these three products, we shall find the product of £358 4s. 7¾d. by 256.

Now, $200 = 10 \times 10 \times 2$; therefore if we multiply the multiplicand by 10, and that product by 10, and this second product by 2, we shall have the third product the same as if we had multiplied the multiplicand at once by 200:

To obtain the product of the multiplicand by 50, we multiply the first product by 10, since $50 = 5 \times 10$: then multiplying the top line by 6, and adding the three products just obtained, we have the product of £358 4s. 7¾d. by 256, viz. £91707 9s. 4d.

£	s.	d.	
358	4	7¾	
		10	
3582	6	5½	= 10 times the top line
		10	
35823	4	7	= 100 times „
		2	
71646	9	2	= 200 times „
17911	12	3½	= 50 times „
2149	7	10½	= 6 times „
£91707	9	4	= 256 times „

52. Similarly, if we had a multiplier of four figures, as 7538, we should have to multiply three times by 10, which would give 1000 times the top line; then by the 7, to make 7000 times.

Also the third line, which equals 100 times the top line, must be multiplied by 5— 500

and the second line, which equals 10 times the top line, must be multiplied by 3— .... 30 „

and the top line itself by 8— .... 8 „

Therefore, the sum of all the four products so formed will be equal to 7538 times.

And similarly for any other number.

53. It seems unnecessary to state a separate Rule for multiplying by such numbers as we have been just using in the above Examples, since the process of multiplication, for forming the products to be added together, has been explained in the Rule already given for multiplying compound quantities by any number not exceeding 12.

Exs. 35.

	£	s.	d.			£	s.	d.	
1.	8	9	7½	× 23	13.	1	17	9¼	× 237
2.	11	19	8	× 39	14.	10	19	11½	× 348
3.	14	13	7¼	× 47	15.	42	3	6¼	× 375
4.	21	2	0¾	× 53	16.	2	7	9¾	× 573
5.	35	17	11	× 61	17.	15	7	9¼	× 943
6.	42	8	8¾	× 75	18.	3	17	7¼	× 1103
7.	101	9	10	× 83	19.	3	0	7½	× 1215
8.	215	7	8¾	× 104	20.	4	9	7½	× 3201
9.	45	5	9¼	× 137	21.	31	16	11	× 4375
10.	178	11	10	× 174	22.	1	11	6	× 7235
11.	216	13	4	× 180	23.	16	16	6½	× 4520
12.	189	6	8	× 196	24.	1	7	11	× 11829

TROY WEIGHT.

	lbs.	oz.	dwts.			lbs.	oz.	dwts.	grs.	
25.	15	9	17	× 35	28.	14	11	19	13	× 137
26.	75	8	13	× 57	29.	45	9	14	16	× 785
27.	117	11	19	× 89	30.	170	8	18	17	× 8923

AVOIRDUPOIS WEIGHT.

	cwts.	lbs.	oz.	drs.			tons	cwts.	lbs.	oz.	
31.	14	75	14	6	× 74	34.	45	17	101	14	× 175
32.	35	54	8	5	× 801	35.	73	14	96	11	× 432
33.	147	108	4	13	× 945	36.	185	11	10	9	× 819

APOTHECARIES' WEIGHT.

	oz.	drs.	sc.	grs.			lbs.	oz.	drs.	sc.	
37.	11	7	2	15	× 215	40.	11	4	7	2	× 574
38.	9	5	1	7	× 307	41.	35	11	6	1	× 809
39.	10	6	0	19	× 95	42.	79	10	5	0	× 199

LONG MEASURE.

	mls.	fur.	po.	yds.			fur.	po.	yds.	ft.
43.	175	6	35	4 × 89		46.	15	38	5	2 × 587
44.	216	5	19	3 × 117		47.	175	17	3	1 × 2017
45.	384	7	27	2 × 438		48.	83	37	4	2 × 7845

CLOTH MEASURE.

	yds.	qrs.	nls.			E. ells	qrs.	nls.	in.
49.	476	3	3 × 73		52.	78	3	2	1 × 487
50.	894	2	1 × 194		53.	805	2	3	2 × 534
51.	1727	1	2 × 216		54.	648	3	2	2 × 1749

ALE AND BEER MEASURE.

	hhds.	gals.	qts.			bar.	gals.	qts.	pts.
55.	114	51	1 × 98		58.	67	15	3	1 × 415
56.	268	47	3 × 375		59.	395	31	2	0 × 1209
57.	897	15	2 × 4021		60.	427	27	3	1 × 877

WINE MEASURE.

	hhds.	gals.	qts.			hhds.	gals.	qts.	pts.
61.	87	59	3 × 65		64.	45	17	2	1 × 407
62.	275	60	2 × 183		65.	306	43	3	0 × 196
63.	349	15	1 × 235		66.	742	37	2	1 × 384

SQUARE MEASURE.

	sq. yds.	sq. ft.	sq. in.			ac.	ro.	sq. po.	sq. yds.
67.	75	5	75 × 73		70.	75	3	25	17 × 117
68.	117	7	108 × 85		71.	127	2	35	25 × 245
69.	237	8	125 × 97		72.	345	1	17	30 × 367

CUBIC MEASURE.

	sol. yds.	ft.	in.			sol. yds.	ft.	in.
73.	83	15	1000 × 35		75.	305	20	584 × 115
74.	215	17	896 × 72		76.	785	25	1425 × 327

WOOL WEIGHT.

	sacks	ods	lbs.			packs	lbs.
77.	45	11	25 × 275		79.	875	215 × 489
78.	138	10	27 × 604		80.	1000	175 × 563

TIME.

	hrs.	min.	sec.			dys.	hrs.	min.	sec.
81.	17	57	45 × 875		83.	119	11	30	17 × 1823
82.	23	45	59 × 904		84.	235	17	37	25 × 1987

COMPOUND DIVISION.

54. Ex. I. To divide £189 8s. 4d. by 8.

The £189 can be divided precisely as in Simple Division, and the remainder is £5. To divide this by 8, bring it to shillings; this gives 100 sh., and with the 8s. in the dividend = 108 sh.: dividing by 8, as in Short Division, we have 13s., and 4s. over. Bring this to pence; it = 48d., and with the 4d. in the dividend = 52d.: dividing by 8, the quotient is 6d., and the remainder 4d.: bring this to farthings; it = 16 farthings, which divided by 8 gives 2 farthings, or 1 halfpenny; and the complete quotient is £23 13s. 6½d.

	£	s.	d.
8)	189	8	4
	£23	13	6½

55. The same alterations that we described in Compound Addition, Subtraction, and Multiplication, are to be made here, when the compound quantity consists of any other than pounds, shillings, pence, and farthings. For instance, if we had for a dividend a quantity consisting of tons, cwts., qrs., &c.; then, after dividing the tons, we should have to reduce the remaining tons to cwts., and add in the cwts. already in the dividend; so also any remaining cwts. would have to be reduced to qrs.; the remaining qrs. to lbs.; and so on.

Hence when we have to divide a compound quantity by a number under 12, we have the following

RULE. Divide the highest denomination as in Short Division; if there be a remainder, reduce it to the next name, and add to it the units there may be of this next name in the dividend, and divide again. Proceed in like manner through all the denominations, treating the remainders after each division, exactly as the first remainder. But if, after any division, there be no remainder, then divide, if possible, the units of the succeeding denomination; but if these units be not equal in number to the divisor, place a cipher underneath as quotient, and treat these units as a remainder to be reduced to the next denomination.

Divide

EXS. 36.

	£	s.	d.	
I.	88	2	6	by 3, 5, 7.
II.	268	3	1½	by 3, 5, 6, 10.
III.	517	11	0	by 4, 6, 8, 12.

TROY WEIGHT.

	lbs.	oz.	dwts.	grs.			lbs.	oz.	dwts.	grs.
1.	18	9	15	16 ÷ 5		4.	117	7	14	20 ÷ 10
2.	175	11	7	21 ÷ 7		5.	359	11	19	6 ÷ 11
3.	308	9	16	5 ÷ 9		6.	1827	10	0	12 ÷ 12

AVOIRDUPOIS WEIGHT.

	cwts.	lbs.	oz.	drs.			tons	cwts.	qrs.	lbs.
7.	75	15	8	15 ÷ 6		9.	45	11	3	20 ÷ 8
8.	315	94	13	12 ÷ 7		10.	117	9	2	17 ÷ 9

57. When the divisor exceeds 12, but is a number which can be exactly formed by the multiplication of two numbers each less than 12, then we can divide by the two numbers successively. Thus, if the divisor be 27, or 9×3, we divide first by the 9 and then by the 3, as was shewn (26) in

Simple Division. Also, the two remainders must be formed into one, as was shewn in the same article.*

Ex. II. Divide £189 8s. 4d. by 27.

	£	s.	d.	
3)	189	8	4	
9)	63	2	$9\frac{1}{4}$	1 } 22 rem^r †
	£7	0	$3\frac{1}{2}$	7 }

Exs. 37. Find the required quotients in the following Examples.

	£	s.	d.			£	s.	d.	
1.	616	17	6	÷ 35	7.	893	17	11	÷ 45
2.	804	9	$4\frac{1}{2}$	÷ 90	8.	1356	8	$10\frac{1}{2}$	÷ 72
3.	1025	2	0	÷ 48	9.	4589	16	$8\frac{1}{4}$	÷ 99
4.	987	6	$4\frac{3}{4}$	÷ 56	10.	8764	14	$10\frac{3}{4}$	÷ 100
5.	594	18	9	÷ 63	11.	5783	19	$1\frac{3}{4}$	÷ 121
6.	1023	7	$6\frac{1}{4}$	÷ 36	12.	6897	11	$0\frac{1}{2}$	÷ 144

LONG MEASURE.

	mls.	fur.	po.	yds.			faths.	yds.	ft.	in.	
13.	15	5	30	3	÷ 27	16.	75	1	2	9	÷ 84
14.	48	6	35	2	÷ 35	17.	117	0	1	10	÷ 90
15.	96	7	13	4	÷ 81	18.	875	1	2	11	÷ 110

TIME.

	yrs.	wks.	dys.	hrs.			yrs.	dys.	hrs	min.	
19.	75	43	3	15	÷ 36	22.	144	14	17	45	÷ 64
20.	96	51	6	17	÷ 49	23.	903	245	14	18	÷ 72
21.	107	18	4	11	÷ 55	24.	1000	75	19	30	÷ 121

57. But when the divisor exceeds 12, and cannot be broken up as in the last article, it is necessary to divide by a process similar to that of Simple Long Division; but as

---

*The learner will see hereafter that the best mode of forming the complete remainder will be—not to have farthings in the quotient, and a remainder besides; but to express as a fraction of a penny the whole of the quantities remaining after the pence in the quotient.

† The more correct form of the quotient, as described in the previous note, will be £7 0s. $3\frac{1}{2}\frac{2}{7}$d.

the only difference between Compound Long Division and Compound Short Division is, that in the former case the reducing of the remainders is performed on the paper, and in the latter case in the memory, we shall not write out a new rule for Long Division, but merely give one Example to shew the method of working.

Ex. III. To divide £3178 11s. 7½d. by 754.

$$
\begin{array}{rrrl}
 & \text{£} & s. & d. \quad \text{£} \quad s. \quad d. \\
754) & 3178 & 11 & 7\tfrac{1}{2} \;(4 \quad 4 \quad 3\tfrac{3}{4} \\
 & \underline{3016} & & \\
 & 162 & & \\
 & \underline{20} & & \\
 & 3251 & (4s. & \\
 & \underline{3016} & & \\
 & 235 & & \\
 & \underline{12} & & \\
 & 2827 & (3d. & \\
 & \underline{2262} & & \\
 & 565 & & \\
 & \underline{4} & & \\
 & 2262 & (3 \text{ farthings} & \\
 & \underline{2262} & & \\
\end{array}
$$

Here we see the division after each reduction by 20, 12, and 4, to be precisely of the same kind as in Simple Division.

Exs. 38. Find the required quotients in the following Examples.

	£	s.	d.			£	s.	d.	
1.	45	18	9	+ 117	11.	1745	9	10½	+ 875
2.	236	17	6½	+ 235	12.	2834	10	7	+ 908
3.	435	9	8¼	+ 384	13.	3479	12	9¾	+ 1423
4.	1027	16	9	+ 379	14.	6842	14	3	+ 1785
5.	1847	11	7¾	+ 645	15.	999	8	5¼	+ 1827
6.	8923	14	6	+ 893	16.	10101	10	4	+ 3459
7.	11023	3	4½	+ 1001	17.	4586	16	6½	+ 6083
8.	8976	18	5	÷ 986	18.	3897	13	8	+ 8756
9.	7643	15	4¼	+ 1237	19.	7854	11	10¾	+ 9802
10.	8962	14	11	+ 1461	20.	6923	7	11	+ 1143

SQUARE MEASURE.

	ac.	ro.	po.	yds.			sq. mls.	ac.	ro.	po.
21.	1175	3	15	15	+ 870	24.	11485	480	3	17 + 785
22.	207	2	20	17	+ 927	25.	9746	523	2	20 + 983
23.	199	1	27	19	+ 1345	26.	10409	610	1	29 + 1425

CUBIC MEASURE.

	sol. yds.	ft.	in.			sol. yds.	ft.	in.
27.	8963	20	1000	+ 3024	29.	15684	19	897 + 2568
28.	11429	15	1684	+ 4837	30.	83746	26	1432 + 11984

31. If 721 persons earn £626 7s. 4¼d., how much is that to each?
32. The expenses of a railway are £10299 1s. 8d. per annum; how much per day?
33. Find the price per ounce of a piece of gold weighing 375¼ oz., and costing £1548 18s. 9d.
34. A person owes £1000, and has only £758 6s. 8d.; how much can he pay for every pound which he owes?

58. There is one other kind of Examples which may be found in Compound Division, viz. when it is required to find how often one compound quantity is contained in another.

Ex. IV. How often is £3 15s. 3¼d. contained in £26 6s. 10¾d.

Here we must reduce both quantities to the same name, farthings: and the question becomes one of Simple Division, viz., How often are 3613 farthings contained in 25291 farthings? and the result is, 7. But such questions are more properly treated under Reduction: and some examples of the kind will be found in the following pages.

REDUCTION.

PART II.

59. I will now give one instance of each kind of the more difficult Examples that involve Reduction and the Compound Rules.

Ex. I. How many times must I take a stride of 2 ft. 9 in., in walking 7 miles ?

This question is similar to the examples in (43) and (44), and in plain language means—How often is the quantity 2 ft. 9 in. contained in 7 miles, or in seven times 1760 yds. ?

Bringing both quantities into portions of 3 inches, I have the annexed work, where I multiply the 2 feet by 4, because there are in one foot four parts each three inches; and the 12320 yards by 12, because there are in one yard 12 parts each three inches.

$$\begin{array}{ll|rl}
 & & \text{yds.} & \\
 & & 1760 & \\
\text{ft.} & \text{in.} & 7 & \\
\cline{3-3}
2 & 9 & 12320 & \text{yds.} \\
4 & & 12 & \\
\cline{1-1}\cline{3-3}
\multicolumn{2}{l}{11 \text{ portions})} & 147840 & \text{portions} \\
\cline{3-3}
 & & 13440 & \text{steps.}
\end{array}$$

Ex. II. How many times will a coach wheel revolve, in going 175 miles, if its circumference be 15½ feet ?

Comparing this question with the previous one, and putting the circumference of the wheel in place of the length of the man's step, the questions are seen to be of the same kind.

The work will be

```
                          miles.
                           175
                          1760
                         10500
                        1225
ft.                     175
15½                     308000 yds.
 2                           6
31 half ft.       31) 1848000 hf. ft. (59612
                      155
                       298
                       279
                        190        Ans. 59612 times, and 28 half ft.
                        186                  or 14 ft. remaining.
                          40
                          31
                           90
                           62
                           28
```

An example the opposite of the two last will be as follows.

Ex. III. A man takes 3744 strides in walking a distance of 1 m. 1516 yds.; what is the length of each step?

I have here merely to divide the whole distance by the number of strides, by Compound Long Division, as follows:

```
              m.  yds.
              1   1516
              1760
              3276 yds.
                 3
      3744) 9828 ft. (2 ft. 7½ in.
            7488
            2340
              12
           28080 (7 in.
           26208
            1872
               2
            3744 (1 hf. in.
            3744
```

The correctness of the answer of this last Example may be proved by working the following question.

How far will a man walk, if he takes 3744 strides, each 2 ft. $7\frac{1}{2}$ inches ?

This question will of course be worked by Compound Multiplication, and the answer will be 1 m. 1516 yds.

Ex. IV. In 55 purses, each containing a guinea, a moidore, a sovereign, a crown, and a groat, how many pounds sterling ?

I must here find the value of the whole of the coins in one purse, and then multiply the sum by 55. The work will be as here shown.

£	s.	d.	
1	1	0	
1	7	0	
1	0	0	
	5	0	
		4	
3	13	4	
		5	$\times 11 = 55$.
18	6	8	= 5 times.
		11	
201	13	4	= 55 times.

Ex. V. How many crowns, half-crowns, shillings, and groats, amount to £99 16s. 4d.; taking of each an equal number ?

The sum of all these coins must be found, and I must then see, by division, how often that sum is contained in the £99 16s. 4d.; the quotient will be the required number of coins of each kind.

The sum of the coins is 8s. 10d.; since therefore my dividend is £99 16s. 4d., and my divisor 8s. 10d., I must reduce both to pence, and the whole work will be as follows ;

s.	*d.*		*£*	*s.*	*d.*	
5	0					
2	6					
1	0		99	16	4	
	4		20			
8	10		1996 sh.			
12			12			
106d.		)	23956d.	(226		
			212			
			275			
			212			
			636			
			636			*Ans.* 226 of each.

question which is the reverse of this would be,
226 purses, each containing a crown, a half-crown, a
g, and a groat, how many pounds? And this ques-
s of the same kind as Ex. IV.

. VI. I changed £27 14s. 6d. for pieces of 15d., 10d.,
nd 4d., taking of each an equal number: how many
h had I?

is question will be found to be the same as V., if I
t in the following form:—How many coins of 15d.,
9d., and 4d., amount to £27 14s. 6d., taking of each
ual number? The work is

d.		*£*	*s.*	*d.*	
15		27	14	6	
10		20			
9		554 sh.			
4		12			
38d.	)	6654d.			(175 pieces, and 4d. remain.
		38			
		285			
		266			
		194			
		190			
		4			

me of the Examples given below will involve various
ts and measures; but they can all be worked upon
rinciples of the questions thus shown. In the first

twelve Examples, two questions are given upon each of the above six Examples in order; but the remainder are mixed, and it must be left to the judgment of the learner to discover which among these six is the mode of working. And it will be a very good exercise for a beginner to endeavour in such questions to find out the reverse question, as I have done in Exs. (III.) and (V.)

Exs. 39.

1. The interval between the tollings of a bell is 7½ seconds, how many times will it be heard in 1 hour 45 min. 30 sec.?
2. I take a stride of 2 ft. 8⅓ in., but put my stick to the ground only every other step; how often will it touch the ground if I walk 17¼ miles?
3. The circumference of a wheel is 5¾ feet; how many times will it turn in 175⅜ miles?
4. The piston of a steam engine travels 18 miles 396 yds. 1 ft. 2 in.; how many times must it have oscillated, if its stroke be 2 ft. 7 in.?
5. What is the circumference of a wheel which revolves 1570 times in 1 mile 28 yds. 2 in.?
6. A bridge, containing 75 arches and 76 piers, measures 1 furlong 156 yds. 2 ft.; what is the length of each arch, if each pier is half the length of an arch?
7. What amount will be obtained from 357 subscribers, each contributing a guinea, a crown, a half-crown, and a shilling?
8. 1175 casks contain each 3 gallons, 3 quarts, 3 pints, and 3 half-pints; how much do they all hold?
9. How many parcels of 1¾ lbs., 2¼ lbs., and 10 lbs., can be obtained out of a cask of sugar weighing 7 cwt. 2 qrs. 11 lbs, taking of each an equal number?
10. A field of 68 acres 3 roods is divided into equal numbers of plots of 1 rood 25 poles, and of 2 roods 35 poles; how many will there be?
11. I exchanged 355 guineas, worth £1 6s. 6d. each, for one-pound notes, crowns, and groats, taking of each an equal number; how many of each had I; and what was their value?
12. An estate of 375 acres, and another of the same size, worth twice as much, was exchanged for equal numbers of allotments of 3 acres, of 2 roods, and of 25 perches, all of the former quality; how many portions should I receive in all?

13. There are 1000 subscribers to a charity, each giving a guinea, a pound, and a crown; and 525, each giving a half-crown, a shilling, and sixpence; how much is raised in all?
14. In the clothing of a regiment, 655 suits are made each containing 3 yds. 1 qr. 3 nls. of cloth, and 1 yd. 3 qrs. 1 nl. of lining; how many yards of material are required for the whole?
15. A certain vessel contains a hogshead, a barrel, a kilderkin, and a quart; how many such can be filled out of 30758 gallons?
16. From 3 cwt. 2 qrs. 14 lbs. I take away 1 lb. 1 oz. and 1 dram; how many times can I do this, and what will remain?
17. How many intervals of 3 minutes, 2 min., $\frac{1}{2}$ min., $\frac{1}{4}$ min., can be made out of a fortnight, and of each an equal number?
18. A man walks up a hill in 7 min. $35\frac{1}{4}$ sec., and down again in 5 min. 13 sec.; how many times can he repeat this in 7 hrs. 2 min. $40\frac{1}{4}$ sec.?
19. A piece of land is required, which can be divided into allotments of 3 roods, 2 roods, 1 rood, 35 perches, and 25 perehes; there are to be 127 of each allotment; how many acres must the land contain?
20. A peal of 10 bells has 2629 changes rung in 3 hours, what is the average length of interval between the stroke of each bell, if all the bells are rung at every change?

Exs. 40.

MISCELLANEOUS EXAMPLES.

1. If a spoon weigh 15 dwts. 11 grs., how many dozen of such spoons can be formed out of 122 oz. 9 dwts. 1 gr.?
2. How many loaves of 4 lb., 2 lb., and $1\frac{1}{2}$ lb. can be made out of 12 sacks of flour, each weighing 240 lbs.?
3. Find the difference in hours between 3 yrs. 10 mths. 3 wks. 6 dys., and 4 yrs. 9 mths. 2 wks. and 4 dys.
4. Find the age of a person whose pulse has beat 589,764,385 times, at the rate of 70 per minute.
5. *A* was born at 10 o'clock p.m., on Sept. 26, 1845; and *B* at 3 a.m., on April 15th, 1846; how much older in hours is *A* than *B*?
6. An estate costing £17897 6s. 4d. is divided into 5876 allotments; what is the value of each share?
7. The diameter of a crown is $1\frac{1}{2}$ inches; how many will it take to reach $11\frac{1}{2}$ miles?

8. Three thousand and eighty-nine subscribers contribute £3 13s. 4d. each, how much will they raise in all?

9. A piece of cloth 500 yds. 2 qrs. in length is cut up to form coats, each requiring 1 yd. 3 qrs. 2 nls.; how many coats can be obtained from it?

10. How many subscribers, of £2 11s. 7½d. each, will be required to purchase an estate worth £11953 15s. 4½d.?

11. A square whose side is 385 inches, is divided into oblongs, 7 inches by 5, how many will there be?

12. How many intervals of 5 minutes 35 seconds are in a leap year?

13. I exchange 4375 yds. for pieces of 3 qrs. 2 nls.; how many did I receive?

14. Find the number of cubic yards in 2,877,580 inches.

15. How often must the sum of 5s. 0½d., 7s. 7d., and £1 17s. 4½d., be repeated to make £200?

16. What would be the length of an acre of ground, if its breadth were 60½ yds.?

17. What is the amount of £3 4s. 4½d. repeated 800 times?

18. Find the difference between £34 15s. 9½d. + £78 7s. 6d. and £135 10s. 9d. – £84 17s. 10½d.

19. The number of solid feet or inches, in a block of wood, having all its sides oblong, is found by multiplying the length, breadth, and thickness; find the number of solid feet in a block 245 inches long, 39 broad, and 17 thick.

20. If 6½ millions of visitors entered the Crystal Palace in 26 weeks, what was the average attendance per day?

21. How many acres in a field 387 yds. long, and 275 yds. broad?

22. Sound travels about 1100 feet per second; what is the interval between the flash and the sound, when the storm is 4 miles off?

23. In 25 bales, each containing 24 pieces, and each piece 45½ yards, how many lengths of 4 yds. 2 qrs. 3 nls.?

24. If a spoon cost 7s. 9½d., how many dozen can be bought for £44 8s. 3d.?

25. A man who owes £2348, pays 12s. 9½d. for every pound which he owes; how much does he pay in all?

26. The cost of gilding is 4½d. per square inch; find the expense of gilding a box, of which the dimensions are 7 in., 9 in. and 15 inches.

27. A book requires 25¼ sheets of paper; how large an edition can be printed from 184 reams 2 quires 7 sheets?

28. Fifteen hundred men contribute £1 3s. 6½d. each, and as many children half as much; how much money is raised?

29. How much per day is 1000 guineas per leap year?

30. If a pint contains 2728 barleycorns, how many will it take to fill a sack?
31. A man spends £155 5s. 7d. per year; how much will he lay by in 37 years, out of £200 per annum?
32. The divisor is £32 6s. 8d., the remainder 10s. 6d., and the quotient is 375; what is the dividend?
33. Among how many persons can I divide £1764 15s., so that they may each have £1 2s. 7½d.?
34. If the sun's light comes to us in 8 min. 58 sec., and the distance is 95,000,000 miles, what is the rate of the light per second?
35. On a railway, the rails weigh 70 lbs. per yard, and the chairs, weighing 14 lbs. each, are placed at intervals of 18 inches, how many tons of iron are employed in making 20 miles of a double line of rails?
36. If 1 lb. Avoirdupois is equal to 14 oz. 11 dwt. 16 grs. Troy, how many lbs. Troy in 6 cwt.?
37. Thirty-five cheeses weigh 17 lbs. 3 oz. each; how many pieces of 4 oz., 6 oz., 1 lb., and 1¼ lbs., can be cut from them, taking of each an equal number?
38. Five bells of different tones are successively struck at intervals of 3 minutes, 2 min., 1 min., $\frac{1}{2}$ min., and $\frac{1}{4}$ min.; how many times could I hear the whole round of tones in 9 hrs. 27 minutes?
39. In the last question, which bell should I have last heard at the end of 4 hrs. 58 min.?
40. How many intervals of 3 min. 35¾ sec. in half a century?
41. A wall is 8½ feet high, 236 feet long, and 17 inches thick; there are in it two doorways each 6 feet by 4 feet; how many bricks would be required for it, each containing 204 solid inches?
42. Divide £357 12s. 2d. among 3 men, 4 women, and 6 children, giving to each man twice as much as to a woman, and four times as much as to a child.
43. A mixture is made of 4 gallons at 3s. 9d., 5 gals. at 4s. 6d., and 11 gals. at 6s. 8d.; what is the value of a gallon of the mixture?
44. A court-yard, 150 ft. square, is surrounded by a walk 24 feet broad; and a grassplot occupies the remainder; find the area of the walk and grassplot.
45. In a foot-race, *A* gains on *B* at the rate of 5 yds. in 1 min. 50 secs., how soon will he be half a mile a-head?

ON THE PRINCIPLES OF THE SIMPLE RULES.

1. What do you mean by Numeration? Give examples of its use?
2. Explain how numbers greater than 10 can be represented, though there are but ten different figures.
3. What do you mean by the term "digits?"
4. Express by signs the addition of the numbers 365, 4000, and 18, with the subtraction of 1728 and 496. Give the result.
5. Describe an Addition Table; and shew how by its aid you add together two quantities, one of which is more, and the other less, than 10.
6. Explain the common process of borrowing employed in Subtraction, and shew what is the correct mode of making allowance for it.
7. How do you prove a sum in Subtraction?
8. Write down in signs—"the sum of 17 and 8 is equal to the difference between 36 and 11."
9. Write down in signs—"the product of 18 and 12 is equal to the quotient of 2156 divided by 11."
10. Shew why you can multiply by 10, 100, 1000, &c. more easily than by any other numbers..
11. If the quotient, remainder, and dividend are known, how will you find a divisor? Construct such an example, and find the divisor.
12. Explain how a Multiplication Table is made.
13. Shew how it can be used as a Division Table.
14. If the floor of an oblong room were covered with square tiles, all of the same size, how would you find the number of them without countiug them all?
15. If you knew how many bricks were used in paving a floor, and how many bricks the floor was in length, how would you find the number in the breadth?
16. Shew that Short Division is the same as Long Division, only that the work is performed mentally.
17. How do you *prove* a Division Sum?
18. What is the meaning of the terms *divisor*, *dividend*, and *quotient*?
19. Explain the process of forming the complete remainder, when you divide by a composite divisor. Ex. Divide 325 shillings among 72 people.
20. Shew how to divide by such numbers as 20, 300, 4000, &c., and explain the correctness of the remainder.

ON REDUCTION AND THE COMPOUND RULES.

1. What do you mean by *Compound* Rules?
2. What is meant by Tables of Weights and Measures? Write out "Long Measure."
3. In a Compound Addition sum, of money, what other numbers may be found besides the whole numbers found in the Simple Rules?
4. Write down *one*-farthing or one-quarter, *two* farthings, *three* farthings, in figures.
5. How otherwise can you write two farthings, and by what other name would you then call it?
6. Explain, without working the questions, the mode of operation in the following cases:
   (1) Reduce 3 cwt. 3 qrs. $17\frac{1}{2}$ lbs. to half lbs.
   (2) Reduce 19275 farthings to pounds.
7. State the steps whereby you would find how many persons could receive each 3s. 4d. out of £20.
8. Explain the mode of multiplying any sum of pounds, shillings, and pence, by 3256.
9. Construct a question, which requires reduction by both processes of multiplication and division; and work it.
10. Explain the mode of multiplying by $5\frac{1}{2}$, and $30\frac{1}{4}$.
11. Shew how to divide by the same quantities.
12. Form an example the opposite to question 11 in Exs. 40.

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A

FAMILIAR EXPLANATION

OF THE HIGHER PARTS OF

RITHMETIC:

COMPRISING

FRACTIONS, DECIMALS, PRACTICE;

PORTION, AND ITS APPLICATIONS;

ETC. ETC.

WITH AN APPENDIX.

DESIGNED AS AN INTRODUCTION TO ALGEBRA.

BY THE

REV. FREDERICK CALDER, B.A.

AD MASTER OF THE GRAMMAR SCHOOL, CHESTERFIELD.

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PREFACE.

issuing a second edition of this work, I am bound ıcknowledge the kind reception which it has met h both from the public press and from teachers. ave received many letters from Tutors and Schoolsters, expressing their opinion that the book has plied a want which had long been felt. I have, refore, endeavoured to make the book more worthy public approbation.

The principal alteration in this Edition is, that the amples, which had before been published only in a arate form, are now inserted in their proper places the text, and are considerably increased in number, that the book is now independent of any other work. e answers, however, have not been appended to the ies intended for general use, but they may be ained, either separately, or bound up with the thmetic, at a corresponding increase in price. In ing the answers of most of the more difficult quesıs, I have put down the most important steps in the ıtion, which I hope will be found of much service to se pupils who cannot meet with efficient assistance.

The Articles on Decimals and the Extraction of Roots have received considerable alterations; and the Chapter on Discount has, I think, been much improved, by the insertion of a valuable rule concerning the relative proportion of the net and gross prices of goods subject to discount, for the principle of which I am indebted to J. F. Ledsam, Esq., of Birmingham.

The subjects of Contracted Decimals and Scales of Notation have also been added, and explained, I think, more fully than is usual in works on Arithmetic.

I have inserted, at various intervals, papers of miscellaneous questions, involving all the rules and operations preceding the points where they are inserted. There are also appended Collections of Questions upon the definitions, principles, and modes of operation adopted in my Arithmetic; for I think it a very valuable exercise that pupils should be required to explain in writing, in as plain language as they can, the definitions and proofs of rules in Arithmetic, in the same manner as is usually expected of those who are reading Algebra.

In a short time I hope to publish the elementary part mentioned in the Preface to the First Edition, and so render the book complete. The parts will be sold either separately, or together.

F. C.

CONTENTS.

THE FOLLOWING SIGNS MUST BE VERY WELL KNOWN BEFORE THIS BOOK IS READ.

$+$ *plus*, placed between two numbers, shows that they are to be added together.

$-$ *minus*, between two numbers, shows that the latter number is to be subtracted from the former.

$\times$ *into*, or *multiplied by*, between two numbers, shows that the two are to be multiplied together.

$\div$ *by*, or *divided by*, between two numbers, shows that the former is to be divided by the latter.

$=$ *equal to*, between two quantities, shows that the quantities on each side of it are equal.

$\therefore$ stands for *therefore*.

$\because$ *because*, or *since*.

$>$ *greater than*, placed between two quantities, shows that the former is greater than the latter.

$<$ *less than*, placed between two quantities, shows that the former is less than the latter.

$\sim$ placed between two quantities, shows that the smaller of the two is to be subtracted from the larger.

Note.—The figures enclosed in parenthesis, as (8) in p. 4, line 22, refer to previous articles in the work.

ARITHMETIC.

Before commencing the consideration of Fractions, it will be necessary to explain certain properties of numbers, which enter very much into the treatment of fractions.

MEASURES AND MULTIPLES.

1. Of two numbers, the larger is said to be a *multiple* of the smaller, when the larger can be divided exactly, *i. e.* without remainder, by the smaller; and the smaller number is said to be a *measure* of the larger. Thus, since 12 can be divided exactly by 4, 12 is called a multiple of 4, and 4 a measure of 12. So, 16 is a multiple of 8, and 8 a measure of 16.

2. Also, one number is said to be a *common* multiple of two or more numbers, when it can be divided exactly by the two or more numbers. Thus, 24 is a common multiple of 6, 4, and 2, because it can be divided exactly by those numbers; and 12 is the *least* common multiple of 6, 4, and 2, because it is the *least* number that can be divided exactly by those three divisors.

3. A number is said to be a common measure of two or more numbers when it will exactly divide those numbers;

and the greatest number which will so divide them is called their greatest common measure. Thus, 2, 4, 6, 12, are common measures of 24 and 36; and 12 is their greatest common measure.

We shall for the future write G. C. M. for greatest common measure, and L. C. M. for least common multiple.

4. A number is called a *prime* number when it cannot be divided exactly by any number greater than 1. Thus, 3, 5, 7, are prime numbers, because they have no divisor greater than 1; we also call such numbers *primes*.

5. Two or more numbers are said to be prime to one another, when they cannot *all* be divided exactly by any number greater than 1. Thus 7, 8, 9, are prime to each other: but such numbers are not necessarily themselves prime numbers; for 8 can be divided by 2 and 4; and 9 can be divided by 3; but 8 and 9 have no divisor common to both.

6. All numbers not prime are called *composite;* because they are composed of 2 or more prime numbers multiplied together. Thus, 6 is a composite number, being formed by the multiplication of the prime numbers 2 and 3.

7. The prime numbers which, when multiplied together, form a composite number, are called *factors* of that number. Thus, 2, 2, 2, 3, are factors of 24, because $2 \times 2 \times 2 \times 3 = 24$. So the factors of 36 are 2, 2, 3, 3, for $2 \times 2 \times 3 \times 3 = 36$.

8. To break up a composite number into its prime factors, divide it by the smallest prime which will divide it without remainder; and continue dividing by prime divisors until the last quotient is 1. The number will be found equal to the product of all the divisors—that is, all the divisors are its factors.

Ex. To break 120 into prime factors:

$$\begin{array}{r|l} 2) & 120 \\ 2) & 60 \\ 2) & 30 \\ 3) & 15 \\ 5) & 5 \\ & 1 \end{array}$$

$\therefore$ 2, 2, 2, 3, 5, are its prime factors, or, $2 \times 2 \times 2 \times 3 \times 5 = 120$.

When, therefore, I say—break any number, as 120, into its factors, I mean, exhibit it in the above form. This form is called an equation: thus, $7 \times 5 = 35$ is an equation; and 7×5 is called the left-hand *side*, and 35 the right-hand *side*, of the equation.

Exs. 1. Resolve into prime factors

1. 1050.	3. 1485.	5. 1820.	7. 64.	9. 2310.	11. 5724.
2. 2625.	4. 1155.	6. 4802.	8. 4389.	10. 6342.	12. 1168.

Obs. If any one number be divisible by a second, we must have all the factors of the second number among the factors of the first: thus, if 36 be divisible by 12, all the factors of 12 will be among the factors of 36, or 36 must contain all the factors of 12.

9. Also, if I wish to divide 35 by 5, (since 35 is the same as 7×5,) I may divide 7×5 by 5; and since I know that the quotient is 7, I can now observe that that quotient is found by taking away the divisor 5 out of the product 7×5: so also in the equation $7 \times 3 \times 5 = 105$, to divide the 105 by 5, take away the 5 out of the factors on the left-hand side, and there remains 7×3, or 21, as quotient; to divide by 7, take away 7, and 3×5, or 15, is quotient; or, we can divide by the product of 2 factors at once: thus, to divide by 15, take away 3×5, and the quotient is 7. This is a very quick method of dividing a composite number by any one or more of its factors, especially if the number be large.

Thus, since $2520 = 2 \times 2 \times 2 \times 7 \times 3 \times 3 \times 5$;

$\text{or} = 8 \times 7 \times 9 \times 5$;

therefore, if I wish to divide 2520 by any measure of it, as 24, I take away 24, or $2 \times 2 \times 2 \times 3$, from its factors, and the product of the remaining factors, $7 \times 3 \times 5 = 105$, gives the quotient. To divide 2520 by 63, I remove 7×9, and the quotient $= 8 \times 5 = 40$.

LEAST COMMON MULTIPLE.

10. It is required to find the L. C. M. of 2, 3, 5, 6, 9, 10, 12, 18, 20.

Now, of the above numbers, we observe that 2, 3, 6, and 9 are divisors, or measures, of 18; every number, therefore, which is a multiple of 18, will also be a multiple of 2, 3, 6, 9. So likewise, every multiple of 20 will be a multiple of 5 and 10; therefore, every multiple of 18 and of 20 will be a multiple of 2, 3, 6, 9, 5 and 10. If, then, I find the L. C. M. of the three remaining numbers 12, 18, and 20, that multiple will be the L. C. M. of 2, 3, 5, 6, 9, 10, 12, 18, 20. Now, $12 \times 18 \times 20$, or 4320 gives *one* common multiple of 12, 18, and 20, since this product is plainly divisible by 12, 18, and 20; but this is not the *least* C. M.

11. To find the L. C. M., break the numbers 12, 18, 20, into their prime factors, as in (8), and place a comma between each set of factors, thus:

12,	18,	20,
$2 \times 2 \times 3$,	$2 \times 3 \times 3$,	$2 \times 2 \times 5$.

Now, by (8 OBS.), in order that the L. C. M. may contain 12, it must contain all the factors of 12; ∴ it must contain 2, 2, 3,

so, to contain 18, ... 2, 3, 3,

and to contain 20, ... 2, 2, 5;

that is, in the required L. C. M. I want two 2's, two 3's, and one 5; but in the product of $12 \times 18 \times 20$, or of $2 \times 2 \times 3 \times 2 \times 3 \times 3 \times 2 \times 2 \times 5$, I have five 2's and three 3's; therefore I have three 2's and one 3 which I do not require; and, omitting these, the remaining factors, multiplied together, give $2 \times 2 \times 3 \times 3 \times 5 = 180$: where it may be seen upon trial that this row of factors contains all that is necessary for 12, 18, and 20, and no more; and therefore the product of these factors is the least number divisible by 12, 18, and 20, or is their L. C. M.

12. The whole work should be written out as follows, where the mark (—) is placed over every number which is afterwards to be omitted :—

$$\overline{2},\ \overline{3},\ \overline{5},\ \overline{6},\ \overline{9},\ \overline{10},\ 12,\ 18,\ 20.$$

$$2 \times 2 \times \overline{3}, \qquad \overline{2} \times 3 \times 3, \qquad \overline{2} \times \overline{2} \times 5,$$

$$\therefore \text{L. C. M.} = 2 \times 2 \times 3 \times 3 \times 5 = 180.$$

Ex. II. To find the L. C. M. of 7, 16, 32, 21, 56, 42,

$$\overline{7},\ \overline{16},\ 32,\ \overline{21},\ 56,\ 42.$$

$$2 \times 2 \times 2 \times 2 \times 2, \qquad \overline{2} \times \overline{2} \times \overline{2} \times \overline{7}, \qquad \overline{2} \times 3 \times 7.$$

$$\therefore \text{L. C. M.} = 32 \times 21 = 672.$$

In this example, I first reject 7 and 21, because 42 contains them; next I reject 16, because 32 contains it; then, out of all the rest, I preserve five 2's for the number 32, and reject the others : I preserve one 7, which is required for 56 and 42, and one 3 for 42. In preserving the five 2's, it is better to keep them all together, rather than have some in one set of factors, and some in another.

Exs. 2. Find the L. C. M. of

1. 2, 3, 4, 8.
2. 3, 5, 9, 16, 20.
3. 8, 11, 5, 35, 21.
4. 7, 28, 35, 42, 63.
5. 4, 5, 8, 24, 40, 120.
6. 5, 7, 9, 12, 15.
7. 13, 14, 56, 63, 72.
8. 1, 2, 3, 4, 5, 6, 7, 8, 9.
9. 12, 33, 55, 27, 18.
10. 7, 11, 13, 17.
11. 8, 9, 10, 11, 12, 15.
12. 10, 14, 21, 28, 35.
13. 18, 20, 24, 36, 48.
14. 27, 36, 45, 42, 16.

GREATEST COMMON MEASURE.

13. We have seen in (3) that the G. C. M. of two or more numbers is the largest divisor which can exactly be contained in those numbers.

Where the numbers are not large, this G. C. M. may be found by breaking them into their prime factors; and if any factors are contained in all the numbers, the product of these will be the G. C. M.

Thus, to find G. C. M. of 36, 27, 144; breaking up into factors, we have—

36,	27,	144,
$2\times 2\times 3\times 3$,	$3\times 3\times 3$,	$2\times 2\times 2\times 2\times 3\times 3$,

Here it is plain that in these sets of factors two 3's, and no other number, are common to all; that is, 3×3, or 9, is common to all the numbers 36, 27, 144,—and is therefore their G. C. M.

14. But where the numbers are large, and they cannot be easily broken into factors, the following rule is to be used:—

Take two of the proposed numbers, and divide the greater by the less: if there be a remainder, make that remainder a new divisor, and take the former divisor as a new dividend, and continue this process until there be no remainder: *the last divisor will be the* G. C. M.

If there be a third number, go through the same work with this third number, and the G. C. M. of the other two; then, as before, the last divisor will be the G. C. M. And the same process must be continued if there be 4, 5, &c., or any amount of numbers, of which we have to find the G. C. M.

We will try this rule upon the three numbers taken above, viz. 36, 27, and 144.

```
27) 36 (1                 9) 144 (16
    27                       9
    ---                      --
     9) 27 (3                54
        27                   54
        --                   --
```

∴ 9 is G. C. M. of 27 and 36; and 9 is G. C. M. of 27, 36, and 144, as was shown above.

Ex. II. Find G. C. M. of 324, 456, 728.

```
324) 456 (1                       12) 728 (60
     324                              720
     ---                              ---
     132) 324 (2                        8) 12 (1
          264                              8
          ---                              --
           60) 132 (2                      4) 8 (2
               120                            8
               ---                            -
                12) 60 (5
                    60
                    --
```

∴ 12 is the G. C. M. of 324, 456; and 4 is the G. C. M. of 324, 456, 728.

15. If the last divisor be 1, we then learn that the numbers have no common divisor greater than 1; *i. e.* they are prime to each other.

OBS. Even when the divisors are less than 12, it is better to divide by long, rather than short division, because the remainders are thereby placed in the most convenient situation for continuing the process. This rule for finding the G. C. M. cannot be proved true without the use of algebra; but the *method* of proof is shewn below.*

---

* When any number measures two others, it will be found to measure any number composed either of the *sum* of any multiples of those numbers, or their *difference*.

For example, since 6 is a common measure of 18 and 24, it will be a measure of $(3 \times 18) + (4 \times 24)$ which $= 150$; or of $(7 \times 18) - (3 \times 24)$ which $= 54$. This rule can be *proved* true by algebra, and may be seen to be correct in the particular Ex. just given.

Making use of this fact, and referring to the first portion of Ex. II., I observe that every C. M. of 324 and 456 must be a measure of $(456 - 324)$ or of 132; also that every C. M. of 132 and 324 must be a measure of $(132 + 324)$, or of 456; hence the common measures of 324 and 456 are precisely the same as those of 132 and 324; and therefore the *greatest* C. M. of 324 and 456 is the same as the *greatest* C. M. of 132 and 324: Again, I see that 60 and 132 are derived from 132 and 324, just as these latter numbers were derived from 324 and 456; hence it is plain that the G. C. M. of 132 and 324 is also the G. C. M. of 60 and 132, and still further that it is the same as the G. C. M. of 12 and 60; hence the G. C. M. of the original numbers 324 and 456 is shewn to be the same as that of 12 and 60; but the G. C. M. of 12 and 60 is 12; hence 12 is the G. C. M. of 324 and 456.

16. When we have to find the L. C. M. or G. C. M. of any numbers which are not large, we may often, after a little experience, *see* the answer, without going through the work.

Thus, if it were required to find L. C. M. of 3, 4, 6, and 8, a pupil would soon learn that 24 was the required number. The easiest method of performing this operation, without writing, is to multiply in one's head the largest of the numbers given, by 2, 3, 4, and so on, until a number be found which will contain all the other numbers. So, in 3, 4, 6, and 8, if I try $2 \times 8 = 16$, it will not hold the 6; but $3 \times 8 = 24$ will hold the 6 as well as the 3 and 4, and therefore is the required L. C. M.

17. Again, to find G. C. M. of 4, 8, 12: it is easy to *see* that no number greater than 4 can go in 4, 8, and 12, or that 4 is their G. C. M.

When we can thus *see* the answer without working, we are said to find the L. C. M. or G. C. M. by *inspection*.

Exs. 3. Find the G. C. M. of

1.	348 and 390.	5.	1836 and 1845.
2.	510 „ 595.	6.	4775 „ 10959.
3.	413 „ 343.	7.	715, 781, and 1067.
4.	217 „ 643.	8.	189, 216, „ 729.

Shew that 327 and 529, also that 189 and 728 are prime to each other.

OBS.—THE FOLLOWING ABBREVIATIONS WILL SOMETIMES BE USED.

Den^r	*for* Denominator.	Rem^r.	*for* Remainder.
Num^r	„ Numerator.	Diff^e	„ Difference.
Fr^n	„ Fraction.	Comp^d	„ Compound.
Mult^n . . .	„ Multiplication.	Red^d	„ Reduced.
Add^n	„ Addition.	Com.	„ Common.
Sub^n . . .	„ Subtraction.	Quan^y . . .	„ Quantity.
Div^n	„ Division.	Imp^r	„ Improper.
Quot^t . . .	„ Quotient.	Frac^l	„ Fractional.
Ex.	„ Example.	Art.	„ Article.

FRACTIONS.

18. Def. By the term *Unit*, we are to understand a single article of any kind, as 1 inch, 1 yard, 1 penny, 1 ounce, &c.

Def. *Unity* is merely another name for the figure 1.

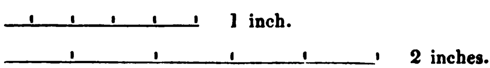
19. A fraction is a part or parts of a number, or quantity, supposed to be broken into any number of equal portions. If, then, the *unit* be divided into 4, 5, or 6 equal parts, one of these parts will be called one-fourth, one-fifth, or one-sixth, and is thus written—$\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$. So also, if any *two* of these parts be taken, the quantities thus taken will be called *two*-fourths, *two*-fifths, and *two*-sixths, and be written $\frac{2}{4}$, $\frac{2}{5}$, $\frac{2}{6}$.

20. The number below the line, which shows into how many parts the *unit* was broken, is called the *denominator*, because it expresses the denomination, or kind of parts, as *fourths, fifths, sixths*. The upper number, which *enumerates*, or counts, how many parts are taken, is called the *numerator*. Thus, in the fraction $\frac{2}{5}$, 5 is the denominator, 2 the numerator.

Obs. Such a fraction as I have been describing is called a *Vulgar Fraction*. We shall afterwards find that *Vulgar* Fractions of a particular class can be expressed in another form, and are then called *Decimal* Fractions, or more commonly *Decimals*.

21. Again, any fraction, as $\frac{2}{5}$, may represent other quantities, besides being two-fifths of *one*, as we have just explained.

Fig. 1.



For if we divide the quantity 1, as 1 inch, into 5 equal parts (see Fig. 1), and also divide the quantity 2 into 5 equal parts, then one of the *latter* parts will be twice as great as one of the *former*; or *two* of the former and *one* of the latter have the same value. Now, the former or smaller is one-fifth of 1; and the latter or larger is one-fifth of 2; and, since two of the former = one of the latter, therefore two-fifths of 1 = one-fifth of 2, or $\frac{2}{5}$ of 1 = $\frac{1}{5}$ of 2; and both these quantities are expressed by the fraction $\frac{2}{5}$. Similarly, $\frac{6}{7}$ means *six*-sevenths of *one*, or *one*-seventh of *six*; $\frac{3}{4}$sh. means three-fourths of 1 shilling, or one-fourth of 3 shillings, each of which will, on trial, be found 9d.

22. From what has been said, it will be seen that the den^r of a fraction shows into how many parts the unit is divided; since, then, $\frac{6}{7}$ is proved to mean one-seventh of 6, that is, it = 6 divided by 7; we hence see that a number placed as a den^r, underneath any other number as a num^r, shows that this den^r is to be taken as a divisor of the num^r; thus, $\frac{15}{7}$ implies that 15 is to be divided by 7: but as long as I do not work out the div^n, that division is said only to be *expressed*. Thus again, in the fraction $\frac{23}{7}$, we understand that 23 *is to be divided* by 7.

We shall hereafter see more clearly than now, that if 23 is divided by 7, the fraction $\frac{23}{7}$ may be called the *quotient*.—(See Appendix, Art., *Fractional Quotient.*)

23. A fraction is called a *proper* fraction when the num^r is less than the den^r; thus $\frac{3}{8}$, $\frac{4}{5}$, are called *proper* fractions, because they really represent a part or parts of a unit, and are less than the whole unit.

24. A fraction whose num^r is equal to, or is greater than the denom^r, is called an *improper* fraction: Thus, $\frac{5}{5}$, $\frac{12}{7}$, are called *improper* fractions, because they are not in

reality *parts* of a unit broken up—*i. e.* are not less than the whole unit—but they are either one complete unit, or more than one.

25. Any whole number may be made to appear as an improper fraction with any required den^r, by multiplying and dividing the whole number by that denominator. Thus, $3 = \frac{3 \times 5}{5} = \frac{15}{5}$: if the required den^r be 7, $3 = \frac{21}{7}$.

26. A mixed number is one formed of a whole number and a fraction, as $3\frac{2}{5}$, which is read three and two-fifths—that is, three units, and two-fifths of another unit; just as 3s. 7d. means 3 shillings and $\frac{7}{12}$ of another shilling, and might be written $3\frac{7}{12}$sh., where the unit is one shilling.

27. A fraction consisting of two or more fractions, connected by the word *of* placed between them, is called a *compound* fraction; as $\frac{2}{3}$ of $\frac{5}{7}$ of $\frac{8}{9}$: and a *complex* fraction is one in which either the num^r or den^r, or both, are fractions; as

$$\frac{2\frac{3}{4}}{4}, \qquad \frac{3}{7\frac{1}{2}}, \qquad \frac{1\frac{1}{3}}{3\frac{2}{5}}, \qquad \frac{\frac{8}{9} \text{ of } 1\frac{2}{3}}{7\frac{1}{5}}, \text{ \&c.}$$

This last fraction is thus read: eight-ninths of one-and-two-thirds, divided by seven-and-one-fifth.

28. To multiply a fraction by any whole number, we multiply the num^r and retain the same den^r.

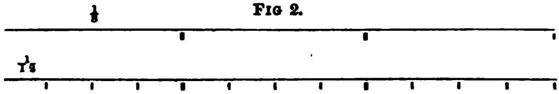
Thus, if it be required to multiply $\frac{5}{12}$ by 4,—just as 4 times 5 *pence* = 20 *pence*, so 4 times *5-twelfths* = 20-*twelfths*, or $\frac{20}{12}$; $\therefore \frac{5}{12} \times 4 = \frac{20}{12}$, which $= \frac{5 \times 4}{12}$; *i. e.* to multiply a fraction by a whole number, the num^r is multiplied by the number, and the den^r is not altered.

29. Again, to multiply a fraction by any whole number, *if possible*, divide the den^r by this multiplier, and leave

the numerator unaltered. By this rule we should have $\frac{5}{12} \times 4 = \frac{5}{12 \div 4} = \frac{5}{3}$.

To prove this, let any unit be divided—1st, into 3 equal parts, and 2ndly, into 12 equal parts (see Fig. 2); then

FIG 2.



each of the larger parts is called $\frac{1}{3}$, and each of the smaller $\frac{1}{12}$. Now, one of the *thirds* is four times as large as one of the *twelfths*, therefore 5 of such thirds are 4 times as large as 5 of the twelfths; *i. e.* $\frac{5}{3} = 4$ times $\frac{5}{12} = 4 \times \frac{5}{12}$; or, in the former shape, $\frac{5}{12} \times 4 = \frac{5}{3}$, which $= \frac{5}{12 \div 4}$; that is, in multiplying by a whole number, the den^r must be divided, and the num^r left unaltered.

I subjoin a specimen of the form in which these Exs. should be worked.

$$\frac{5}{18} \times 6 = \frac{5}{18 \div 6} = \frac{5}{3}.$$

$$\frac{5}{18} \times 8 = \frac{5 \times 8}{18} = \frac{40}{18}.$$

EXS. 4.

I. Multiply $\frac{5}{6}$ by 2, 3, 4, 5, 6, 7, 8, 9, successively.

II. Multiply $\frac{19}{24}$ by 3, 5, 6, 8, 9, 12, successively.

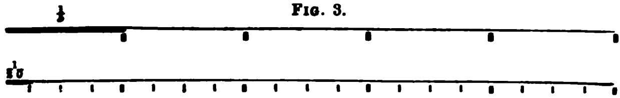
30. To divide a fraction by any whole number, divide the num^r, if possible, and keep the same den^r.

Thus, if it be required to divide $\frac{12}{13}$ by 4, just as 12 *pence* divided by 4 would give a quot^t 3 *pence,* so 12 *thirteenths* divided by 4, would give a quot^t 3 *thirteenths;* or $\frac{12}{13} \div 4 = \frac{3}{13}$ which $= \frac{12 \div 4}{13}$; *i. e.* the num^r is divided by the given divisor, and the den^r unaltered.

But, if we cannot divide the num^r of the fr^n by the whole number, then we must multiply the den^r by this divisor. By this method $\frac{3}{5} \div 4$ would $= \frac{3}{5 \times 4} = \frac{3}{20}$.

To prove this, observe in the fractions $\frac{3}{5}$ and $\frac{3}{20}$, that in the first fr^n the unit is divided into 5 equal parts, (see Fig. 3,) and therefore each $= \frac{1}{5}$; and, in the second, the same unit is divided into 20 equal parts, and therefore each $= \frac{1}{20}$:

FIG. 3.



now it is plain, from Fig. 3, that *one*-fifth, when divided by 4, becomes *one*-twentieth; so *three*-fifths divided by 4 become *three*-twentieths; *i. e.* $\frac{3}{5} \div 4 = \frac{3}{20}$, which $= \frac{3}{5 \times 4}$; hence we multiply the den^r by the given divisor, and leave the num^r unaltered.

The following are specimens of the mode of working.

$$\frac{9}{10} \div 3 = \frac{9 \div 3}{10} = \frac{3}{10}.$$

$$\frac{9}{10} \div 6 = \frac{9}{10 \times 6} = \frac{9}{60}.$$

OBS. Since it rarely happens that we can divide any num^r or den^r by any chance number, we therefore, in multiplying a fr^n, have, as a general rule, to multiply the num^r; and in dividing a fr^n, to multiply the den^r.

EXS. 4.

III. Divide $\frac{8}{9}$ by 2, 3, 4, 5, 6, 7, 8, 9, successively.

IV. Divide $\frac{36}{49}$ by 2, 3, 4, 7, 8, 9, 18, successively.

31. We have now shown that to multiply a fr^n by a whole number, we multiply the *num^r*, and to divide a fr^n we multiply the *den^r;* therefore if we multiply both num^r and

den^r by the same quan^y, we shall have both multiplied and divided the fr^n by the same number; but since to multiply a quan^y by any number, and then to divide by the same, leaves the original quan^y unaltered, therefore to multiply both num^r and den^r of a fr^n will not alter its value :

$$\text{Thus, } \frac{3}{5} = \frac{3 \times 4}{5 \times 4} = \frac{12}{20};$$

and by observing Fig. 3, we shall see that since 3 of the large divisions, or *fifths*, $\doteq$ 12 of the small divisions, or *twentieths*, we have by inspection the same result, viz. $\frac{3}{5} = \frac{12}{20}$.

32. So also, since by dividing the num^r we *divide* the whole fr^n, and by dividing the den^r we *multiply* the same fr^n; therefore, if we divide both num^r and den^r by the same quantity, the fr^n will remain unaltered in value :

$$\text{Thus, } \frac{12}{20} = \frac{12 \div 4}{20 \div 4} = \frac{3}{5};$$

by a process opposite to that in the last Art.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

(**Thrower, Case I.**)

33. We have shown that both num^r and den^r of a fr^n can be divided by the same number without altering the value of the fr^n; and if the num^r and den^r cannot be exactly divided by any whole number, the fr^n is said to be in its lowest terms, because it is expressed in the lowest possible numbers.

Thus, $\frac{3}{4}$, $\frac{5}{6}$ are fr^{ns} in their lowest terms, because 3 and 4 in the first fr^n, and 5 and 6 in the second fr^n, have no common divisor, or C. M.

But $\frac{8}{12}$ is not in its lowest terms, for both num^r and den^r can be divided by 4; and the fr^n will then be $\frac{2}{3}$. So also, in fr^n $\frac{15}{25}$, 5 is G. C. M. of num^r and den^r; and the fr^n in its lowest terms $= \frac{3}{5}$.

It is plain, therefore, that if any fr^n be not in its lowest terms, and we wish so to reduce it, both num^r and den^r must be divided by the greatest possible divisor, *i. e.* by their G. C. M.

In working a sum of this kind we should express the operation thus:

$$\frac{15}{25} = \frac{15 \div 5}{25 \div 5} = \frac{3}{5}$$

but when, in future, it is necessary to use this operation, it may be thus written, $\frac{15}{25} = \frac{3}{5}$, the intermediate step being omitted.

34. When the num^r and den^r of a fr^n are both small numbers, the G. C. M. may be often found by *inspection*, or sometimes *some* C. M. may be found, though not the greatest; and, by dividing by it, the fr^n may *partly* be reduced; and then a C. M. of the fr^n so reduced may be found, and a second division performed, till we reduce the fr^n if not to the lowest, at any rate to lower terms.

Thus, $\frac{35}{245} = \frac{7}{49} = \frac{1}{7}$ is now reduced to lowest terms.

It is worth notice that every number ending in 5 or 0, must have 5 for a divisor, and every even number can be divided by 2.

But if mere inspection will not tell us the G. C. M. of num^r and den^r, it must be found by the method already given (14) for finding the G. C. M. of two given numbers.

Exs. 5. Reduce to their lowest terms

1. $\frac{24}{40}$	3. $\frac{275}{425}$	5. $\frac{1197}{2471}$	7. $\frac{985}{1379}$	9. $\frac{459}{1560}$	11. $\frac{343}{539}$
2. $\frac{85}{153}$	4. $\frac{1368}{3096}$	6. $\frac{2241}{2700}$	8. $\frac{5705}{6559}$	10. $\frac{1111}{9090}$	12. $\frac{891}{540}$

TO REDUCE AN IMPROPER FRACTION TO A WHOLE OR MIXED NUMBER.

(Thrower, Case II.)

35. Ex. To reduce $\frac{17}{5}$ to a whole or mixed number. We have before shown (22) that this fr^n expresses that 17 is to be divided by 5: now, if the 17 were exactly divisible by 5, the quotient would be a whole number; but if not, as in this case, then since 15 is the largest multiple of 5 below 17, we can divide the 15 by 5, and the quotient is 3; but the division of the remaining 2 by the 5 cannot be *performed:* it must therefore be *understood,* by placing the divisor 5 underneath the 2 as a den^r, and thus $\frac{2}{5}$ may be said to be the quot^t of 2 when divided by 5: the whole quotient will, therefore, be 3 and $\frac{2}{5}$; and the operation may be thus written—

$$\frac{17}{5} = \frac{15+2}{5} = \frac{15}{5} + \frac{2}{5} = 3 + \frac{2}{5}$$

or, as it is generally written, $= 3\frac{2}{5}$.

Also, since 5-fifths = 1, therefore 15-fifths = 3, and 17-fifths must = 3 and 2-fifths, or $3\frac{2}{5}$, as just shown.

36. In writing the whole operation as fully as has been done above, it is necessary to find what is the rem^r after the num^r has been divided by the den^r. It must then be placed as the rem^r 2 has been placed in the above Ex., and the 15 is of course found by subtracting this rem^r from the whole num^r. Where the num^r and den^r are small, this rem^r may be found without working any div^n sum on the paper; but if the numbers are large, a div^n sum must be worked before the above operation can be shown.

Thus: Ex. 2. Reduce $\frac{327}{19}$ to a whole or mixed number.

$$\begin{array}{r l}
19)\ 327 & (17\tfrac{4}{19} \\
19 & \\
\hline
137 & \\
133 & \\
\hline
4 & \\
\hline
19 &
\end{array}$$

Here the rem^r being 4, the largest multiple of 19 below 327 is 323, which contains 19, 17 times,

therefore we have $\frac{327}{19} = \frac{323 + 4}{19} = \frac{323}{19} + \frac{4}{19} = 17\tfrac{4}{19}$;

or, briefly written, $\frac{327}{19} = 17\tfrac{4}{19}$.

Exs. 6. Reduce to whole or mixed numbers

1. $\frac{24}{7}$	3. $\frac{288}{15}$	5. $\frac{12211}{113}$	7. $\frac{10001}{99}$	9. $\frac{1759}{176}$	11. $\frac{1749}{55}$
2. $\frac{82}{11}$	4. $\frac{603}{28}$	6. $\frac{5576}{17}$	8. $\frac{9999}{100}$	10. $\frac{8431}{844}$	12. $\frac{4001}{87}$

TO REDUCE A MIXED NUMBER TO AN IMPROPER FRACTION.

(Thrower, Case III.)

37. Ex. To reduce $3\frac{2}{5}$ to an improper fr^n. That we may explain this operation properly, we must observe, that in order to express two or more numbers in one sum, these must all be of one kind or denomination; thus, in a simple addition sum, we add units to units, tens to tens, &c.; and in a compound addition sum we add together like quantities, as pounds to pounds, shillings to shillings, &c.

If, therefore, I have to express in one sum 3 units and 2 fifths, these two quantities must be reduced to the same kind. Now, I cannot reduce the fifths to a whole number, because two-fifths are less than unity; I must, therefore, reduce the 3 units to fifths, that is, I must express 3 as a fr^n with a den^r 5.

Thus, by (25) $3 = \frac{3 \times 5}{5} = \frac{15}{5}$; adding then the 15-fifths to the two-fifths, I have the sum 17-fifths, or $\frac{17}{5}$.

The whole operation may be thus expressed:

$$3\tfrac{2}{5} = \frac{3 \times 5 + 2}{5} = \frac{15 + 2}{5} = \frac{17}{5}.$$

38. It will be seen that this Ex. is exactly the reverse of that in (35); and we may observe, that the num^r of the improper fr^n, viz. 17, is obtained by multiplying the whole number 3 by the den^r 5, and adding the former num^r 2: thus, in working the above Ex., I should say 5 times 3 are 15 and 2 are 17.

When, then, we have in future Ex^s to reduce a mixed number, $3\tfrac{2}{5}$, to an improper fr^n, we should merely write $3\tfrac{2}{5} = \tfrac{17}{5}$, where all the intermediate work can be performed mentally, when the numbers are not large. We could thus write $9\tfrac{7}{12} = \tfrac{115}{12}$, because 12 times 9 are 108 and 7 are 115.

Exs. 7. Reduce to improper fractions

1. $1\tfrac{6}{7}$	3. $13\tfrac{2}{13}$	5. $130\tfrac{43}{114}$	7. $175\tfrac{1}{175}$	9. $145\tfrac{10}{111}$
2. $4\tfrac{1}{8}$	4. $276\tfrac{5}{29}$	6. $100\tfrac{161}{199}$	8. $45\tfrac{121}{343}$	10. $77\tfrac{7}{90}$

TO REDUCE FRACTIONS TO A COMMON DENOMINATOR.

(Thrower, Case IV.)

39. It has been shown (37) that numbers, whether whole or fractional, cannot be added together without being reduced to the same denom^n or kind.

We must, therefore, show how to reduce fractions to a common den^r; and then they will be of the same denomination.

$$\text{Ex. } \frac{3}{4},\ \frac{5}{6},\ \frac{7}{8},\ \frac{9}{10}.$$

Now, in order that the fractions, when reduced, may still consist of as small numbers as possible, we shall always find their *least* common den^r.

Also, since we can alter the form of fractions only by

multiplying or dividing both num^r and den^r, we must therefore multiply these present den^{rs}, each one by some number, so that they all shall be changed into the new den^r; and for this purpose the new den^r must be divisible by all these den^{rs}, and therefore must be their L. C. M.

Working according to (12) we find the L. C. M. of 4, 6, 8, 10, to be 120. We now choose such multipliers as shall make 4, 6, 8, 10, become 120: these will be found by dividing 120 by the 4, 6, 8, 10; and are 30, 20, 15, 12.

Now, we know (31) that if we multiply the den^r of a fr^n by any number, we must also multiply the num^r by the same number, or else the fr^n will be altered in value. We therefore work thus:

$$\frac{3}{4} = \frac{3 \times 30}{4 \times 30} = \frac{90}{120}$$

$$\frac{5}{6} = \frac{5 \times 20}{6 \times 20} = \frac{100}{120}$$

$$\frac{7}{8} = \frac{7 \times 15}{8 \times 15} = \frac{105}{120}$$

$$\frac{9}{10} = \frac{9 \times 12}{10 \times 12} = \frac{108}{120}$$

and these fractions may be thus written, $\frac{90,\ 100,\ 105,\ 108,}{120}$ showing at once that they have the same den^r.

40. In working sums of this kind, we should have to write down the process of finding the L. C. M. of all the den^{rs}, unless we could see it by *inspection*, which cannot be often done, when the numbers are large.

If it can be done, we must then merely write as the first line of the work L. C. M. = 120, or any number that it may chance to be.

Exs. 8. Reduce to their least common denominator

1. $\frac{13}{18}, \frac{4}{5}, \frac{7}{9}, \frac{11}{40}, \frac{15}{16}$	4. $\frac{3}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}$	7. $\frac{7}{15}, \frac{27}{35}, \frac{5}{72}, \frac{51}{63}, \frac{29}{54}$
2. $\frac{1}{7}, \frac{3}{11}, \frac{5}{21}, \frac{9}{56}, \frac{3}{14}$	5. $\frac{2}{3}, \frac{4}{5}, \frac{7}{8}, \frac{11}{12}, \frac{83}{120}$	8. $\frac{14}{17}, \frac{1}{11}, \frac{11}{18}, \frac{17}{33}$
3. $\frac{11}{15}, \frac{13}{16}, \frac{14}{17}, \frac{17}{18}$	6. $\frac{3}{4}, \frac{13}{20}, \frac{11}{18}, \frac{27}{70}, \frac{41}{90}$	9. $\frac{11}{12}, \frac{59}{60}, \frac{17}{24}, \frac{1}{120}$

TO ADD TOGETHER ANY NUMBER OF FRACTIONS.

(Thrower, Case V.)

41. These may be all proper fractions, or some of them may be proper, some improper, and some mixed numbers.

We have already seen that in order to add fractions together, we must reduce them to a com. den^r^. Now, this has been done in the last two Art^s^; and therefore we have only to conclude the operation in the last Ex. by performing the operation of adding the fractions so prepared. Now, we observe, that whatever number of quantities of any kind are to be added together, the *kind* or denomination is not thereby changed: thus, 7 *months* added to 5 *months* would be 12 *months*; so, 4-*twelfths* + 3-*twelfths* = 7-*twelfths*, which, expressed in figures, gives $\frac{4}{12} + \frac{3}{12} = \frac{7}{12}$; hence it is plain that we have only to add the num^rs^ together, and retain the *same* den^r^; therefore, so doing in the last Ex., we have

$$\text{sum of the fr}^{\text{ns}} = \frac{90+100+105+108}{120} = \frac{403}{120}$$
$$= 3\tfrac{43}{120} \text{ by (35).}$$

If the proper fr^n^ $\frac{43}{120}$ had not been in its lowest terms, it must have been reduced to that state before the sum could be called complete.

42. If some of the fr^ns^ to be added together be mixed numbers, we must first add the frac^l^ parts precisely as above; and to this sum then add the amount of the whole numbers. Thus, if the Ex. which we have just worked had been as follows, $3\frac{3}{4} + \frac{5}{6} + 6\frac{7}{8} + 15\frac{9}{10}$ (where the insertion of the whole numbers 3, 6, and 15, is the only alteration), we should then have, as before,

$$\text{sum of the fractional parts} = 3\tfrac{43}{120};$$

therefore, adding in the whole numbers, we have the whole sum $= 3 + 6 + 15 + 3\frac{43}{120} = 27\frac{43}{120}$.

We will now work a second example completely.

Find the sum of $5\frac{2}{3} + 1\frac{3}{4} + \frac{4}{7} + 4\frac{5}{6} + \frac{7}{8}$.

$$\overline{3},\ \overline{4},\ 7,\ 6,\ 8,$$

$$7,\ \overline{2} \times 3,\ \ 2 \times 2 \times 2,$$

$$\text{L.C.M.} = 7 \times 3 \times 8 = 21 \times 8 = 168.$$

$$\frac{2}{3} = \frac{2 \times 56}{3 \times 56} = \frac{112}{168}$$

$$\frac{3}{4} = \frac{3 \times 42}{4 \times 42} = \frac{126}{168}$$

$$\frac{4}{7} = \frac{4 \times 24}{7 \times 24} = \frac{96}{168}$$

$$\frac{5}{6} = \frac{5 \times 28}{6 \times 28} = \frac{140}{168}$$

$$\frac{7}{8} = \frac{7 \times 21}{8 \times 21} = \frac{147}{168}$$

therefore, sum of frac[l] parts $= \dfrac{112 + 126 + 96 + 140 + 147}{168}$

$= \dfrac{621}{128} = 3\frac{117}{168} = 3\frac{39}{56}$ [in lowest terms.]

and whole sum $= 5 + 1 + 4 + 3\frac{39}{56} = 13\frac{39}{56}$.

Obs. If there be any improper fractions in the proposed example, they may be reduced to mixed numbers before we begin to reduce to L. C. D.; or if they are not very large, we may proceed with them as with the proper fractions.

Exs. 9. Find the value of

1. $\frac{3}{4} + \frac{4}{5} + \frac{5}{12} + \frac{11}{16}$
2. $\frac{7}{9} + \frac{5}{8} + \frac{11}{50} + \frac{14}{27}$
3. $\frac{5}{9} + \frac{7}{8} + \frac{11}{16} + \frac{13}{20} + \frac{1}{2}$
4. $\frac{3}{5} + \frac{17}{18} + \frac{4}{25} + \frac{7}{27}$
5. $10\frac{3}{8} + 1\frac{5}{12} + \frac{7}{10} + 4\frac{13}{16}$
6. $17 + 1\frac{3}{11} + \frac{5}{7} + 13\frac{15}{28}$
7. $5\frac{7}{11} + 3\frac{8}{9} + \frac{15}{22} + 17\frac{3}{35} + 1\frac{7}{10}$
8. $7\frac{1}{12} + 8\frac{1}{13} + 9\frac{1}{14} + 10\frac{1}{15}$

SUBTRACTION.

(**Thrower, Case VI.**)

43. The examples under this head will be of different kinds.

1st. When there are two frns, both proper, as $\frac{3}{10}-\frac{1}{4}$, I.

2nd. When there are two frns, one or both of which are mixed numbers, as $7\frac{3}{4}-3\frac{2}{5}$, II.; $8\frac{2}{5}-2\frac{3}{7}$, III.; $8-2\frac{5}{7}$, IV.

If there be any examples containing improper frns, such examples may either be worked as I.; or, by reducing the improper frns to mixed numbers, we bring them under II., III., or IV.

The difference between II., III., and IV. consists in this, that

$$\begin{array}{llll} \text{in II. the former frac}^{l}\text{ part} & \frac{3}{4} \text{ is} > & \text{the latter } \frac{2}{5}; & \\ \text{in III.} \quad ,, \quad\quad ,, \quad\quad ,, & \frac{2}{5} \text{ is} < & \quad ,, \quad \frac{3}{7}; & \text{(A)} \\ \text{in IV.} \quad ,, \quad\quad ,, \quad\quad ,, & \text{is } 0, \text{ and} \therefore < \frac{5}{7}. & & \end{array}$$

44. Under the head of Subtraction will also come examples in which there are more than two fractions; and wherein we have to find the difference between the sum of one set of fractions, and the sum of another set, as in the following

$$\text{Ex. } 5\tfrac{2}{9}+4\tfrac{3}{8}-\tfrac{1}{2}+3\tfrac{5}{6}-7\tfrac{3}{7};$$

wherein the signs + and − show that I am to take both $\frac{1}{2}$ and $7\frac{3}{7}$, that is, the sum of $\frac{1}{2}$ and $7\frac{3}{7}$, from the sum of $5\frac{2}{9}+4\frac{3}{8}+3\frac{5}{6}$.

DEF. If any numbers, either whole or fracl, are joined together by one or more of the signs +, −, &c., the whole set of numbers thus connected is often termed an *expression*.

45. Just as it it has been shown that fractions cannot be added together until they are reduced to a C. D., so it will

appear that one fr^n cannot be subtracted from another till the two fr^{ns} are reduced to a C. D. Thus, if I had to take 3 pence from 1 shilling, I must bring the 1s. to pence, and then take 3d. from 12d., obtaining a rem^r 9d.

Taking, then, Ex. 1. $\frac{3}{10}-\frac{1}{4}$, we may see, by *inspection*, that L.C.D. $=20$;

$$\text{therefore } \frac{3}{10}=\frac{3\times 2}{10\times 2}=\frac{6}{20}; \quad \frac{1}{4}=\frac{1\times 5}{4\times 5}=\frac{5}{20};$$

$$\text{and } \quad \frac{3}{10}-\frac{1}{4}=\frac{6-5}{20}=\frac{1}{20}. \qquad \text{(B)}$$

When the L. C. D. is not large, say under 150, a pupil will, with a little experience, soon be able at once to write down the line (B), without working the two previous operations on paper.

46. It was said in line (A) $\frac{2}{5}<\frac{3}{7}$; a beginner cannot see this at once; but he may do so very readily—thus: we already know that no two quantities can be compared, to see which is the larger, unless they be of the same kind; so, two fr^{ns} cannot be compared unless reduced to a C. D. *Any* C. D. will, however, do for the purpose of merely telling which of the two fr^{ns} is the larger; now the product of the two den^{rs} is always *one* C. D., but not always the *least*; therefore since 35 is a C. D. of $\frac{2}{5}$ and $\frac{3}{7}$,

$$\frac{2}{5}-\frac{3}{7}=\frac{2\times 7-5\times 3}{5\times 7}=\frac{14-15}{35}; \quad \text{(C)}$$

$$\text{and since } 14<15, \text{ therefore } \frac{14}{35}<\frac{15}{35} \text{ or } \frac{2}{5}<\frac{3}{7}.$$

Now this result may be very rapidly obtained without any work on paper; for if we observe line (C), we notice that the numbers 14 and 15, which numbers alone I wish to compare, are found by multiplying the num^r of the 1st frn^n by the den^r of the 2nd, and the den^r of the 1st by the

num^r^ of the 2nd. Thus, beginning with the num^r^ of the 1st, I say, *mentally*, $2 \times 7 = 14$: and again, $3 \times 5 = 15$; therefore, since the first product is less than the second, so also the first fr^n^ is less than the second.

Taking one or two more Exs. of this process—

1. To compare $\frac{3}{7}$ and $\frac{2}{9}$: I say, $3 \times 9 = 27$; $7 \times 2 = 14$; therefore the former fr^n^ is the larger.

2. To compare $\frac{3}{8}$ and $\frac{7}{12}$: I say, $3 \times 12 = 36$; $8 \times 7 = 56$; therefore the latter fr^n^ is the larger.

It will be noticed that the C. D. to which we have just now mentally reduced $\frac{3}{8}$ and $\frac{7}{12}$ is 96; but when we come to the *actual sub^n^*, we shall not use *this* C. D., but the L. C. D. 24.

47. In Exs. II., III., IV., we must employ the same process that we use in Comp^d^ Sub^n^. Take as patterns the three following:

A		B		C	
s.	*d.*	*s.*	*d.*	*s.*	*d.*
18	9	18	2	18	0
15	3	15	5	15	5
3	6	2	9	2	7

Now, in these Exs. it will be seen, that where the number of pence in the upper line exceeds that in the lower line, as in (A), we can at once subtract the 3d. from the 9d., and the 15s. from the 18s., giving a complete rem^r^ 3s. 6d. So in Ex. II., $7\frac{3}{4} - 3\frac{2}{5}$, since $\frac{3}{4} > \frac{2}{5}$, therefore I can take the $\frac{2}{5}$ from the $\frac{3}{4}$, and the 3 from the 7.

The difference of the frac^l^ parts $= \frac{3}{4} - \frac{2}{5} =$ (when reduced to L. C. D.) $\frac{15-8}{20} = \frac{7}{20}$; and diff^e^ of the whole numbers

$= 7 - 3 = 4$; therefore difference of the complete fractions $= (7 - 3) + (\frac{3}{4} - \frac{2}{5}) = 4 + \frac{7}{20}$,

$$= 4\tfrac{7}{20}.$$

The following Ex. is written as it should be worked.

Find the value of $9\frac{3}{8} - 5\frac{1}{5}$.

L. C. D. = 40.

$$\therefore\ \frac{3}{8} - \frac{1}{5} = \frac{15-8}{40} = \frac{7}{40}$$

And $9 - 5 = 4$.

$$\therefore\ 9\tfrac{3}{8} - 5\tfrac{1}{5} = 4\tfrac{7}{40}.$$

48. But in (B) and (C) the number of pence in the lower line is larger than in the upper, and in both cases I must borrow 1s. or 12d. from the 18s. in the upper line, and add this 12d. to the pence in the upper line, if there be any; and when I have sub[d] the 5d. from the 14d. in (B), leaving rem[r] 9d., and from 12d. in (C), leaving rem[r] 7d., I subtract 15s. from 17s., (not 18s., because I have just borrowed 1s. from the 18s.) and in both cases I have a rem[r] 2s.; therefore, the whole rem[r] in (B) is 2s. 9d., and in (C) is 2s. 7d.

I now apply this process to Exs. III. and IV., viz. $8\frac{2}{5} - 2\frac{3}{7}$, III.; and $8 - 2\frac{4}{7}$, IV. As, then, in the Comp[d] Sub[n] I borrowed 1s. from the higher den[n] 18s., and added it to the pence in the upper row, if there were any, so I borrow 1 from the 8, and add it to the former fr[n], if there be any; thus in III. I add the 1 to the $\frac{2}{5}$, and then subtracting $\frac{3}{7}$, I say—

$$1\frac{2}{5} - \frac{3}{7} = \frac{7}{5} - \frac{3}{7} = \frac{49-15}{35} = \frac{34}{35},$$

then, taking the 8 as 7, I have $7 - 2 = 5$; therefore, combining the results of the two subtractions, I have

$$8\tfrac{2}{5} - 2\tfrac{3}{7} = 7 - 2 + \frac{34}{35} = 5 + \tfrac{34}{35} = 5\tfrac{34}{35}.$$

The following Ex. is written as it should be worked.

Find the value of $17\frac{1}{2} - 11\frac{5}{6}$.

$$\frac{1}{2} < \frac{5}{6} \quad \therefore \text{ borrowing 1, I have } 1\frac{1}{2} - \frac{5}{6} = \frac{3}{2} - \frac{5}{6} = \frac{9-5}{6} = \frac{4}{6} = \frac{2}{3}.$$

$$\text{And } 16 - 11 = 5.$$

$$\therefore 17\tfrac{1}{2} - 11\tfrac{5}{6} = 5\tfrac{2}{3}.$$

49. In Ex. IV., $8 - 2\frac{5}{7}$, I borrow 1 from the 8, but as I have no former fr^n to which I may add this 1, as in Ex. III., therefore subtracting the 2nd fr^n $\frac{5}{7}$ from the 1, I have

$$1 - \frac{5}{7} = (\text{when red}^d \text{ to L. C. D.})\ \frac{7}{7} - \frac{5}{7} = \frac{7-5}{7} = \frac{2}{7}; \quad \text{(D)}$$

and taking 7 instead of 8, as before, I have $7 - 2 = 5$; therefore, I should work as follows—

$$1 - \frac{5}{7} = \frac{7}{7} - \frac{5}{7} = \frac{2}{7}.$$

$$\text{And } 7 - 2 = 5.$$

$$\therefore 8 - 2\frac{5}{7} = 5\frac{2}{7}.$$

50. In the line (D) I may observe that when the number 1 is expressed as a fr^n, its num^r is made the same as the den^r of the fr^n which I have to subtract: thus, 7 is the new num^r, and is also the den^r of the original fr^n; whether, therefore, I subtract the smaller num^r from the new num^r or from the den^r of the smaller fr^n, I obtain the same result, viz. a rem^r 2; and I might say at once $1 - \frac{5}{7} = \frac{2}{7}$, where the 2 has been found by subtracting the num^r 5 from its own den^r 7. Working thus, I should have written the whole Ex. as follows:

$$8 - 2\tfrac{5}{7} = (7 - 2) + (1 - \tfrac{5}{7}) = 5\tfrac{2}{7},$$

only that in common use I should not write the two parentheses.

51. Lastly: when there are more than two fractions, as in

Ex. V. $5\frac{2}{9} + 4\frac{3}{8} - \frac{1}{2} + 3\frac{5}{6} - 7\frac{3}{7}$.

Here, omitting for the present the whole numbers, I have to subtract the sum of ^{1} and $\frac{3}{7}$ from the sum of $\frac{2}{9} + \frac{3}{8} + \frac{5}{6}$: since the two frns to be subtracted are together less than the three from which I have to subtract them, no difficulty will occur, and I proceed as in an Ex. in Addition, taking care to arrange the frns to be subtracted last:

$$\frac{2}{9}+\frac{3}{8}+\frac{5}{6}-\frac{1}{2}-\frac{3}{7};$$

$$9,\ 8,\ 6,\ \overline{2},\ 7.$$

$$3 \times 3,\quad 2 \times 2 \times 2,\quad \overline{2} \times \overline{3},\quad 7.$$

$$\text{L.C.D.} = 9 \times 8 \times 7 = 72 \times 7 = 504.$$

$$\frac{2}{9} = \frac{2 \times 56}{9 \times 56} = \frac{112}{504} \qquad \text{See Art. (9).}$$

$$\frac{3}{8} = \frac{3 \times 63}{8 \times 63} = \frac{189}{504}$$

$$\frac{5}{6} = \frac{5 \times 84}{6 \times 84} = \frac{420}{504}$$

$$\frac{1}{2} = \frac{1 \times 252}{2 \times 252} = \frac{252}{504}$$

$$\frac{3}{7} = \frac{3 \times 72}{7 \times 72} = \frac{216}{504}$$

$$\therefore\ \frac{2}{9}+\frac{3}{8}+\frac{5}{6}-\frac{1}{2}-\frac{3}{7} = \frac{112+189+420-252-216}{504}$$

$$= \frac{721-468}{504} = \frac{253}{504}$$

therefore, bringing in the whole numbers with their proper signs,

$$\text{the whole expression} = 5+4+3-7+\frac{253}{504} = 12-7+\frac{253}{504}$$

$$= 5\tfrac{253}{504}.$$

52. We will take one more example of this kind.

Ex. VI. $3\frac{4}{5} - 4\frac{1}{2} + 7\frac{2}{7} - \frac{3}{4}$.

Now here $\frac{1}{2}+\frac{3}{4}$ are $>\frac{4}{5}+\frac{2}{7}$; but since this cannot be seen by mere *inspection,* I proceed as in Ex. V., as though I did not know this, and provide for the difficulty at the proper time. Placing the frns to be subtracted last, I have

$$\frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}$$

$$5,\ 7,\ \overline{2},\ 4,$$

$$\text{L. C. D.}=5\times 7\times 4=140.$$

$$\frac{4}{5}=\frac{4\times 28}{5\times 28}=\frac{112}{140}$$

$$\frac{2}{7}=\frac{2\times 20}{7\times 20}=\frac{40}{140}$$

$$\frac{1}{2}=\frac{1\times 70}{2\times 70}=\frac{70}{140}$$

$$\frac{3}{4}=\frac{3\times 35}{4\times 35}=\frac{105}{140}$$

$$\text{therefore } \frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}=\frac{112+40-70-105}{140}$$

$$=\frac{152-175}{140} \qquad \text{(E)}$$

Since, then, $175>152$, it now appears that I ought to have borrowed 1; but rather than begin the work again, I proceed thus: from the $\frac{152}{140}$ in line (E) I take as much of the $\frac{175}{140}$ as I can, viz. $\frac{152}{140}$, and there will remain $\frac{23}{140}$, which cannot be subtracted, and before which I place the sign ($-$), to show that it has yet to be subtracted; but if I now borrow 1, I can subtract this $\frac{23}{140}$ from the 1, and remember to count the first (+) whole number in Ex. VI. (*i. e.* the 3) as one less than it now stands. The whole operation may be thus written, re-commencing with the line before (E):

$$\frac{4}{5}+\frac{2}{7}-\frac{1}{2}-\frac{3}{4}=\frac{112+40-70-105}{140}$$

$$=\frac{152-175}{140}=\frac{-23}{140}.$$

therefore borrowing the 1, $1-\frac{23}{140}=\frac{140}{140}-\frac{23}{140}=\frac{117}{140}$

and writing 2 for 3 in Ex. VI., and inserting the whole numbers with their proper signs,

the whole expression $=2+7-4+\frac{117}{140}=9-4+\frac{117}{140}$

$=5\frac{117}{140}$.

Exs. 10. Exhibit in one term each of the following expressions:

1. $\frac{5}{7}-\frac{2}{3}$
2. $\frac{22}{45}-\frac{5}{18}$
3. $\frac{29}{56}-\frac{5}{14}$
4. $\frac{15}{16}-\frac{11}{48}$
5. $\frac{25}{36}-\frac{13}{64}$
6. $\frac{18}{19}-\frac{17}{18}$
7. $5\frac{3}{8}-4\frac{2}{7}$
8. $17\frac{9}{10}-11\frac{5}{12}$
9. $8\frac{1}{2}-4\frac{3}{5}$
10. $7\frac{3}{4}-3\frac{11}{12}$
11. $8-3\frac{7}{9}$
12. $17-\frac{1}{2}$
13. $38-27\frac{11}{16}$
14. $3\frac{2}{5}+4\frac{3}{4}-7\frac{1}{2}$
15. $18\frac{1}{4}-7\frac{11}{12}+\frac{5}{9}$
16. $7\frac{1}{2}+4\frac{5}{6}-\frac{7}{12}+3\frac{1}{10}-5\frac{11}{16}$
17. $8\frac{4}{5}-6\frac{3}{7}-\frac{8}{9}+15\frac{1}{3}-3\frac{4}{27}$
18. $18\frac{1}{3}-5\frac{2}{7}+4\frac{8}{15}-\frac{2}{9}-\frac{5}{12}$

MULTIPLICATION.

(**Thrower, Cases VII. and IX.**)

53. We will show, as we proceed, that the operation of multiplication in fractions may be expressed in two ways, and that the word *of* and the sign × placed between frns, have the same meaning: thus, we have to prove that

$\frac{5}{7}$ of $\frac{2}{3}$ of $3\frac{3}{8}$ and $\frac{5}{7}\times\frac{2}{3}\times 3\frac{3}{8}$ have the same value. (F)

The pupil must here clearly call to mind the method of multiplying and dividing fractions by whole numbers. (31)

I will first begin with a pair of fractions, as $\frac{5}{7}\times\frac{2}{3}$.

It has been proved that $\frac{2}{3}$ means *one-third* of 2; therefore I have now to multiply $\frac{5}{7}$ by one-third of 2: hence the product ought to be one-third of the product found by multiplying by 2 alone: *i. e.* if I multiply by 2, and then take one-third of that product, I shall obtain the correct result.

common to both num^r and den^r, these should be struck out before we complete the multiplication.

We see in this case that there is a factor 5 in the num^r and den^r; also, that 18 in the num^r and 3 in the den^r have a common divisor 3: dividing by these common factors, the expression stands thus $\dfrac{\overset{1}{\cancel{5}} \times 2 \times \overset{6}{\cancel{18}}}{7 \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{5}}}$.

The remaining factors in the nnm^r are 1, 2, 6; and in den^r are 7, 1, 1; these form a fraction $\dfrac{2 \times 6}{7} = \dfrac{12}{7} = 1\frac{5}{7}$. This process of striking out factors from num^r and den^r is called *cancelling*, and is to be continued as long as any factors can be found common to both num^r and den^r. When the numbers are large, this cancelling cannot be readily performed, except by those who have had some experience.

Obs. A factor 1 left in a num^r or den^r will not alter the value of the other factors which remain, when we multiply together all those in the num^r, and all those in the den^r; and hence the figures 1 need not be written, as in the above Ex.; but it must be noticed, that if no other factors remain, the value of the num^r or den^r will consist of the product of as many figures 1 as there were factors in the num^r or den^r before cancelling; *i. e.* it will be $= 1$. If, therefore, all the factors in any num^r or den^r have been cancelled, the num^r or den^r of the fraction after cancelling will be 1. But when this occurs in the den^r, it need not be expressed, but the result written as a whole number.

$$\text{Ex. II.}\quad 8\tfrac{2}{5} \times 35 \times \tfrac{17}{24} = \frac{\overset{7}{\cancel{42}}}{\cancel{5}} \times \frac{\overset{7}{\cancel{35}}}{1} \times \frac{17}{\underset{4}{\cancel{24}}}$$

$$= \frac{7 \times 7 \times 17}{4} = \frac{833}{4} \quad \text{(K)}$$

$$= 208\tfrac{1}{4}.$$

Here 42 and 24 had a C. M. 6, and 5 and 35 had a C. M. 5.

The fr^{n} $\frac{833}{4}$ in line (L) will be in its lowest terms, if all the factors common to num^{r} and den^{r} have been cancelled at the proper time.

$$\text{Ex. III. } \tfrac{7}{8} \text{ of } 7\tfrac{1}{2} \text{ of } \tfrac{4}{15} \text{ of } \tfrac{2}{7} = \frac{\cancel{7}}{\underset{2}{\cancel{8}}} \times \frac{\cancel{15}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{15}} \times \frac{\cancel{2}}{\cancel{7}}.$$

Here there are no factors to be *seen* in the num^{r}; but we must remember that in reality there have been 4 divisions of the num^{r} by the factors 7, 15, 4, and 2; and therefore 4 quot^{ts}, which in this case are each = 1; therefore the real num^{r} = 1 × 1 × 1 × 1, or 1; so the quot^{ts} in the den^{r} are 2, 1, 1, 1, and their product = 2; therefore the reduced fraction has 1 as num^{r} and 2 as den^{r}, *i. e.* it is $\frac{1}{2}$.

Exs. 11. Find the value of

1. $\frac{2}{3} \times \frac{4}{5}$ of $\frac{6}{7}$
2. $1\frac{3}{5} \times \frac{4}{27} \times 2\frac{7}{9}$ of $\frac{18}{19}$
3. $3\frac{2}{7}$ of $5\frac{4}{9}$ of $\frac{15}{23}$
4. $10\frac{3}{8} \times \frac{17}{24}$ of $\frac{15}{19}$ of $\frac{13}{34}$
5. $2\frac{6}{7}$ of $\frac{14}{15}$ of $5\frac{1}{3}$ of $7\frac{7}{8}$
6. $\frac{1}{2}$ of $\frac{1}{7}$ of $\frac{1}{9}$ of $4\frac{1}{2}$
7. $3\frac{2}{5} \times 5\frac{1}{2} \times \frac{7}{9} - \frac{1}{3}$ of $\frac{5}{12}$
8. $1 - \frac{3}{11}$ of $\frac{5}{9}$ of $\frac{18}{25}$

DIVISION.

(Thrower, Cases VIII. and X.)

58. The operation of Division may be expressed in two ways;

$$\text{Thus: } \frac{11}{12} \div \frac{5}{9}, \quad \text{and} \quad \frac{\;\frac{11}{12}\;}{\frac{5}{9}}$$

have the same meaning, both being read $\frac{11}{12}$ divided by $\frac{5}{9}$.

We have now to inquire how to perform this division.

We know by (31) that $5 = 9 \times \frac{5}{9} = 9$ times $\frac{5}{9}$, or that 5 is 9 times as large as $\frac{5}{9}$. If, now, I divide $\frac{11}{12}$ by 5 (when my object was to divide by $\frac{5}{9}$,) I shall be dividing by a quantity 9 times too large, and therefore my quotient will be 9 times too small; to obtain, then, the *true* quotient, I

must multiply the first quot^t by 9 ; *i. e.* the correct quot^t is found by performing two operations—first, by dividing by 5, and secondly, by multiplying by 9; and performing these two operations at one step, we have

$$\frac{11}{12} \div \frac{5}{9} = \frac{11}{12} \times \frac{9}{5}$$

and by comparing the two sides of this equation, we see that the divisor $\frac{5}{9}$ has been inverted into $\frac{9}{5}$, and the $(\div)$ has been changed into $(\times)$, or the div^n changed into multiplication.

If mixed or whole numbers are found in any expression which is to be simplified, we must, of course, reduce to imp^r fr^{ns}, and invert the divisor, as before. Thus

$$\frac{9}{14\frac{5}{8}} = \frac{\frac{9}{1}}{\frac{117}{8}} = \frac{9}{1} \times \frac{8}{117} = \frac{\cancel{9}}{1} \times \frac{8}{\underset{13}{\cancel{117}}} = \frac{8}{13}.$$

59. The following is a more complicated Ex. involving both Mult^n and Division.

Reduce to the simplest form $\dfrac{7\frac{3}{5}}{8\frac{5}{9}}$ of $\dfrac{6\frac{5}{12}}{8\frac{5}{6}}$.

Here we reduce mixed numbers to imp^r fr^{ns}—change *of* into $(\times,)$ and invert the divisors, as was shown in last art.; and then, making two steps in the operation,

$$\text{the expression} = \frac{\frac{38}{5}}{\frac{77}{9}} \times \frac{\frac{77}{12}}{\frac{53}{6}} = \frac{38}{5} \times \frac{9}{77} \times \frac{77}{12} \times \frac{6}{53} \quad \text{(L)}$$

$$= \frac{\overset{19}{\cancel{38}}}{5} \times \frac{9}{\cancel{77}} \times \frac{\cancel{77}}{\underset{2}{\cancel{12}}} \times \frac{\cancel{6}}{53} \quad \text{(M}$$

$$= \frac{19 \times 9}{5 \times 53} = \frac{171}{265}.$$

In working such Exs. I should not write down both (L) and (M), but perform the cancelling on the line (L), which would then become (M): but a pupil would have been confused at first, if he had not seen both lines written in full.

Exs. 12. Find the value of

1. $\frac{11}{12} + \frac{2}{3}$
2. $\frac{3}{14} + \frac{1}{7}$
3. $9 + \frac{4}{5}$
4. $18\frac{1}{2} + \frac{6}{7}$
5. $\dfrac{15\frac{3}{4}}{9}$
6. $\dfrac{11}{17\frac{8}{11}}$
7. $\frac{1}{2}$ of $\dfrac{3\frac{2}{3}}{16\frac{1}{9}}$
8. $\dfrac{18}{\frac{3}{5} \text{ of } 4\frac{3}{7}} \div 17\frac{1}{2}$
9. $\dfrac{1}{2} + \dfrac{1}{3 + \dfrac{1}{1\frac{1}{2}}}$
10. $\dfrac{1\frac{1}{12}}{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}$

SIMPLIFICATION OF FRACTIONS.

(Thrower, Case XI.)

60. Under this head is classed a variety of *expressions* intended to exemplify the operations previously illustrated; and the working of the examples under it depends entirely upon a correct and ready use of the methods of performing Addition, Subtraction, &c., as already explained. There is no difficulty in any of them, so long as a pupil will pay close attention to the *signs;* but I will point out one or two Exs. wherein he is liable to err, though every such error must arise entirely from want of proper observation.

Ex. $4\frac{4}{11}$ of $7\frac{5}{16} + 2\frac{1}{2}$ of 11.

In working this, a pupil must take care that he reduces the whole expression on each side of the (+) to the simplest form before he begins to perform the operation of Add^{n}, which that sign (+) bids him perform. The same caution would be necessary if there had been (−) instead of (+) in the above example.

I will work this Ex. as it ought to be written; and leave a large space on each side of the (+), so that I may run no risk of connecting the two parts incorrectly.

$$4\tfrac{4}{11} \text{ of } 7\tfrac{5}{16} \quad + \quad 2\tfrac{1}{2} \text{ of } 11$$

$$= \frac{\overset{3}{\cancel{48}}}{11} \times \frac{117}{\cancel{16}} \quad + \quad \frac{5}{2} \times \frac{11}{1}$$

$$= \quad \frac{351}{11} \quad + \quad \frac{55}{2} \qquad [\text{L. C. D.} = 22]$$

$$= \quad \frac{702 + 605}{22} = \frac{1307}{22} = 59\tfrac{9}{22}.$$

Ex. II. Simplify $\dfrac{3\frac{1}{2} \text{ of } \frac{5}{8}}{2\frac{5}{6} \text{ of } 3\frac{1}{3}} \div \dfrac{2\frac{2}{11} \text{ of } \frac{11}{12}}{3\frac{1}{9} \text{ of } 7\frac{5}{7}}$.

Now, the whole of the fr^{n} to the right of the sign ($\div$) is a divisor of the fr^{n} to the left of the sign; therefore this right-hand fr^{n} must be inverted, and the sign ($\times$) changed to ($\div$): the expression will then be

$$\frac{3\frac{1}{2} \text{ of } \frac{5}{8}}{2\frac{5}{6} \text{ of } 3\frac{1}{3}} \times \frac{3\frac{1}{9} \text{ of } 7\frac{5}{7}}{2\frac{2}{11} \text{ of } \frac{11}{12}};$$

and this, by changing *of* into $\times$, and the mixed numbers into imp^{r} fr^{ns}, becomes

$$\frac{\frac{7}{2} \times \frac{5}{8}}{\frac{17}{6} \times \frac{10}{3}} \times \frac{\frac{28}{9} \times \frac{54}{7}}{\frac{24}{11} \times \frac{11}{12}}$$

and lastly,—inverting the fr^{ns} in the lower line, because they are divisors, and connecting all the fr^{ns} with mult^{n} signs, I have

$$\text{the entire expression} = \frac{7}{2} \times \frac{5}{8} \times \frac{6}{17} \times \frac{3}{10} \times \frac{28}{9} \times \frac{54}{7} \times \frac{11}{24} \times \frac{12}{11} \qquad \text{(N)}$$

$$= \frac{\cancel{7}}{\cancel{2}} \times \frac{\cancel{5}}{\underset{4}{\cancel{8}}} \times \frac{\overset{3}{\cancel{6}}}{17} \times \frac{3}{\underset{2}{\cancel{10}}} \times \frac{\overset{\cancel{14}\,7}{\cancel{28}}}{\cancel{9}} \times \frac{\overset{\cancel{6}\,3}{\cancel{54}}}{\cancel{7}} \times \frac{\cancel{11}}{\underset{2}{\cancel{24}}} \times \frac{\cancel{12}}{\cancel{11}} \; *$$

$$= \frac{3 \times 3 \times 7 \times 3}{4 \times 17} = \frac{189}{68} = 2\tfrac{53}{68}.$$

In working such Exs. I should omit line (N) for the reason given in (59).

I subjoin an Ex., wherein the whole quantity enclosed in ( ) parentheses, or brackets, as they are called, must be reduced to as simple a form as possible, before the ($\times$) or ($\div$) joining the expression in brackets can be made use of.

Thus; if I have to simplify $\left(\frac{7}{8} - \frac{3}{10}\right) \times \left(2\frac{3}{4} + 3\frac{1}{9}\right)$, I shall first reduce $\frac{7}{8} - \frac{3}{10}$ to the simplest form; then $2\frac{3}{4} + 3\frac{1}{9}$ must be reduced in like manner, and these two results must then be multiplied together, because the sign ($\times$) is placed between the brackets. The whole work will stand thus;

$$\frac{7}{8} - \frac{3}{10} = \frac{35 - 12}{40} = \frac{23}{40} \qquad \text{I.} \qquad (\text{since L. C. D.} = 40)$$

$$\text{and } 2\tfrac{3}{4} + 3\tfrac{1}{9} = 2 + 3 + \tfrac{3}{4} + \tfrac{1}{9}$$

$$= 5 + \frac{27 + 4}{36} = 5 + \frac{31}{36} = 5\tfrac{31}{36} \qquad \text{II.}$$

* A pupil will not necessarily perform the cancelling in exactly the same order as above; but the result will be the same.

Therefore, multiplying I. and II., I have

$$\left(\frac{7}{8}-\frac{3}{10}\right)\times\left(2\tfrac{3}{4}+3\tfrac{1}{9}\right)=\frac{23}{40}\times5\tfrac{31}{36}$$
$$=\frac{23}{40}\times\frac{211}{36}$$
$$=\frac{4853}{1440}=3\tfrac{533}{1440}.$$

Exs. 13. Reduce to the simplest forms

I. $\frac{3}{11}$ of $8\frac{2}{3}-\frac{5}{7}$; $\quad\frac{3}{11}$ of $\left(8\frac{2}{3}-\frac{5}{7}\right)$; $\quad\frac{3}{11}\sim\left(8\frac{2}{3}\text{ of }\frac{5}{7}\right)$

II. $\frac{2}{3}$ of $\frac{6}{7}+\frac{8}{9}$ of $17\frac{1}{2}$; $\quad\frac{8}{17}+\left(\frac{3}{5}\text{ of }7\frac{1}{2}\right)+\frac{9}{11}$; $\quad\left(\frac{8}{17}+\frac{3}{5}\text{ of }7\frac{1}{2}\right)+\frac{9}{11}$

III. $\dfrac{2\frac{2}{3}\times\left(3\frac{1}{2}+\frac{7}{9}\right)}{\frac{3}{5}\sim\frac{2}{3}}$; $\quad\dfrac{7\frac{1}{4}\times3\frac{1}{3}}{6\frac{1}{2}\text{ of }\frac{11}{13}}\div\dfrac{\frac{5}{8}\text{ of }11}{\frac{1}{7}}$

IV. $\dfrac{\frac{3}{5}}{\frac{8}{9}}+\dfrac{\frac{5}{7}}{7\frac{1}{2}\text{ of }\frac{11}{12}}$; $\quad\left(5\frac{2}{3}+3\frac{1}{4}\right)\times\left(6\frac{1}{7}\text{ of }\frac{1}{56}\right).$

REDUCTION OF FRACTIONS.

(Thrower, Case XII.)

Def. Any quantity involving money, weight, &c.,—as 4 shillings, 5 pence, $\frac{3}{4}$ oz., is called a *concrete* quantity; and any number not involving any such denominations,—as 6, 8, $\frac{4}{5}$, is called an *abstract* number.

61. It is here intended to express in positive terms such quantities as $\frac{5}{8}$s., $\frac{11}{12}$ of 10s. 6d. &c.; *i. e.* to express concrete quantities, being frac^{l} parts of any given den^{n}, in terms of lower den^{ns}.

Thus, $\frac{5}{8}$s. must be expressed in terms of pence, and frac^{l} parts of a penny; $\frac{11}{12}$ of 10s. 6d. in terms of shillings, pence, and frac^{l} parts of 1d.

So, also, $\frac{9}{11}$ of a ton would be expressed in cwts., qrs., lbs., oz., drs., and frac^{l} parts of a dram.

It may happen that the proposed fr^{n} can be expressed in an exact number of units of some one of the lower den^{ns};

then, of course, there will be no frac^{l} part of the last named den^{n}.

62. I will now show how to perform the operation intended in the last article.

It has been seen (21) that $\frac{5}{8}$s. means either *one*-eighth of 5 shillings, or *five*-eighths of 1 shilling. We may, therefore, use either of these two methods :—divide the 5s. by 8, as in Comp^{d} Short Division, and the process will be thus—

$$\begin{array}{r|l} 8 & \underline{5 \quad 0} \\ & \underline{\underline{\quad 7\tfrac{1}{2}\text{d.}}} \end{array}$$

or, treating the fr^{n} as $\frac{5}{8}$ of 1s., we may change *of* into ($\times$), reduce the 1s. to pence, and proceed thus—

$$\frac{5}{8} \text{ of } 1\text{s.} = \frac{5 \times 12}{8}\text{d.} = \frac{5 \times \overset{3}{\cancel{12}}}{\underset{2}{\cancel{8}}}\text{d.} = \frac{15}{2}\text{d.} = 7\tfrac{1}{2}\text{d.}$$

This latter method seems more suitable to an Ex. which professes to belong to fr^{ns}, and is the one which I recommend, unless the den^{r} of the proposed fr^{n} be very large, so as to make the division to be performed rather difficult: in that case, to work by Comp^{d} Long Div^{n} is preferable, as being more likely to be correct. I will take such an Ex. and work it through by both methods.

Express in positive terms $\frac{17}{26}$ of a ton.

Working fractionally, we have

$$\frac{17}{26} \text{ of } 1 \text{ ton} = \frac{17 \times \overset{10}{\cancel{20}}}{\underset{13}{\cancel{26}}} \text{ cwts.} = \frac{170}{13} \text{ cwts.} = 13\tfrac{1}{13} \text{ cwts.}$$

I now take the frac^{l} part of the cwts., viz. $\frac{1}{13}$ of 1 cwt., and express it in the next lower den^{n}, viz. quarters, and so proceed, till either there be no frac^{l} part left, or till I come to the last den^{n} in Avoirdupois weight.

$\frac{1}{13}$ of 1 cwt. $= \frac{1 \times 4}{13}$ qrs. $= \frac{4}{13}$ qrs. (*i.e.* 0 qrs. in the final quotient.)

$\frac{4}{13}$ of 1 qr. $= \frac{4 \times 28}{13}$ lbs. $= \frac{112}{13}$ lbs. $= 8\frac{8}{13}$ lbs.

$\frac{8}{13}$ of 1 lb. $= \frac{8 \times 16}{13}$ oz. $= \frac{128}{13}$ oz. $= 9\frac{11}{13}$ oz.

$\frac{11}{13}$ of 1 oz. $= \frac{11 \times 16}{13}$ drs. $= \frac{176}{13}$ drs. $= 13\frac{7}{13}$ drs.

$$\text{therefore,} \quad \frac{17}{26} \text{ of 1 ton} = \overset{\text{cwts.}}{13} \ \ \overset{\text{qrs.}}{0} \ \ \overset{\text{lbs.}}{8} \ \ \overset{\text{oz.}}{9} \ \ \overset{\text{drs.}}{13\tfrac{7}{13}}.$$

Secondly, take $\frac{1}{26}$ of 17 tons; *i.e.* divide 17 tons by 26.

$$\begin{array}{rrl}
 & \overset{\text{tons.}}{17} & \\
 & 20 & \\
26) & \overline{340} & (13 \text{ cwt.} \\
 & 26 & \\
 & \overline{80} & \\
 & 78 & \\
 & \overline{2} & \\
 & 4 & \\
26) & \overline{8} & (0 \text{ qrs.} \\
 & 28 & \\
26) & \overline{224} & (8 \text{ lbs.} \\
 & 208 & \\
 & \overline{16} & \\
 & 16 & \\
26) & \overline{256} & (9 \text{ oz.} \\
 & 234 & \\
 & \overline{22} & \\
 & 16 & \\
26) & \overline{352} & (13\tfrac{7}{13} \text{ drs.} \\
 & 26 & \\
 & \overline{92} & \\
 & 78 & \\
 & \overline{14} & \\
 & \frac{14}{26} = \frac{7}{13} &
\end{array}$$

$$\text{therefore, as before,} \quad \frac{17}{26} \text{ tons} = \overset{\text{cwts.}}{13} \ \ \overset{\text{qrs.}}{0} \ \ \overset{\text{lbs.}}{8} \ \ \overset{\text{oz.}}{9} \ \ \overset{\text{drs.}}{13\tfrac{7}{13}}.$$

63. When Exs. under this Case involve money, it is well to carry the reduction no farther than pence and frac[l] parts of a penny: for, since farthings are themselves written as frac[l] parts of a penny, therefore if we carry the reduction to

frac[l] parts of a farthing, the answer will appear somewhat complicated. I will work an Ex. which will illustrate this.

Express in positive terms $\frac{151}{240}$sh.

$$\frac{151}{240}\text{s.} = \frac{151 \times \overset{1}{\cancel{12}}}{\underset{20}{\cancel{240}}}\text{d.} = \frac{151}{20}\text{d.} = 7\tfrac{11}{20}\text{d.} \qquad \text{(O)}$$

$$\frac{11}{20}\text{d.} = \frac{11 \times \overset{1}{\cancel{4}}}{\underset{5}{\cancel{20}}}\text{far.} = \frac{11}{5}\text{f.} = 2\tfrac{1}{5}\text{f.}$$

$$\text{therefore } \frac{151}{240}\text{s.} = 7\tfrac{1}{2}\text{d. } \tfrac{1}{5}\text{f.} \qquad \text{(P)}$$

If I compare $7\frac{11}{20}$d. (O) and $7\frac{1}{2}$d. $\frac{1}{5}$f. in (P), I see that the former fr[n] involves less labour, and conveys to a person acquainted with fr[ns] the idea of the real value of the expression, at least as well as the other.

64. When a unit of any den[n] can be divided exactly by any number, as 4, 5, 6, &c., such fourths, fifths, sixths, &c., are called *aliquot* parts of that unit.

Thus, since £1 or 20s. when divided by 3, 4, 5, 6, 8, 10, 12, gives quotients involving an exact number of shillings and pence; therefore thirds, fourths, fifths, &c., of £1 are called aliquot parts of £1.

For example

$\frac{1}{5}$£ = 4s. $\quad$ $\frac{2}{5}$£ = 8s. $\quad$ $\frac{4}{5}$£ = 16s.

$\frac{1}{8}$£ = 2s. 6d. $\quad$ $\frac{5}{8}$£ = 12s. 6d. $\quad$ $\frac{7}{8}$£ = 17s. 6d.

$\frac{1}{3}$£ = 6s. 8d. $\quad$ $\frac{2}{3}$£ = 13s. 4d.

So also, $\frac{1}{8}$sh. = $1\frac{1}{2}$d. $\quad$ $\frac{5}{8}$sh. = $7\frac{1}{2}$d. $\quad$ $\frac{7}{8}$sh. = $10\frac{1}{2}$d.

And since 1 lb. avoirdupois, or 16oz. can be divided exactly by 2, 4, 8; therefore halves, fourths, and eighths of 1 lb. can be at once expressed in oz.

Thus, $\frac{3}{8}$lb. = 6oz. $\quad$ $\frac{7}{8}$lb. = 14oz.

and if the value of one or more of these aliquot parts of the denominations

in common use be remembered, the operations performed in this Case will often be much shortened.

$$\text{Ex. } \frac{27}{160}£ = \frac{27 \times \cancel{20}}{\underset{8}{\cancel{160}}}\text{sh.} = \frac{27}{8}\text{sh.} = 3\tfrac{3}{8}\text{sh.} = 3\text{s. } 4\tfrac{1}{2}\text{d.}$$

Exs. 14. Express in positive terms

I. $\frac{2}{3}$ of 10s.; $\frac{7}{8}$ of 27s.; $\frac{3}{7}$ of $\frac{4}{15}$ of 19s. 6d.

II. $\frac{9}{11}$ of 1 ton; $\frac{5}{16}$ of a cub. ft.; $\frac{7}{15}$ of a quarter of corn.

III. $\frac{5\frac{1}{3}}{7\frac{1}{2}}$ of $\frac{7}{8}$ of a hhd. of wine; $\frac{5}{11}$ of 1 week; $\frac{7}{16}$ of $365\frac{1}{4}$ days.

IV. $\frac{3}{8}$ of $\frac{5}{9}$ of a Fr. ell; $\frac{1}{11}$ of $\frac{9}{11}$ of $\frac{1}{7\frac{1}{4}}$ of a square mile.

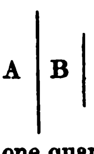
V. $\frac{7}{15}$ of a sq. yd.; $2\frac{1}{2}$ of $\frac{5}{9}$ of a lb. (Apoth.)

RATIO AND PROPORTION.

65. The next operation in Fractions will consist of the expression of one quantity in terms of, or as a frac[1] part of, another of like nature. But before proceeding to attempt this operation, it will be advisable to discuss one of the most important relations in numbers, without which a pupil cannot understand the principle upon which such Exs. will be worked. This relation is termed RATIO.

And as this consideration of Ratio leads to the doctrine of PROPORTION, I have thought it well to continue the subject in one unbroken thread, and then refer to this explanation, when I come to the treatment of questions involving Proportion and its applications.

66. When two quantities are placed before us—as, for instance, the lines A, B—and we wish to compare their



magnitude, we perceive that there is a certain relation between them, which we familiarly call the comparative size of them; and this comparison or relation is measured by seeing how often the one quantity contains the other.

Now, to see how often one quan^{y} contains the other—as, for example, how often 36 contains 5—we divide the former by the latter, *i. e.* the 36 by the 5, and the quotient is $\frac{36}{5}$; therefore, to see how often the line A contains the line B, I divide A by B : *i. e.* the relation of A to B is expressed by the fr^{n} $\frac{A}{B}$.

This relation is called the *ratio* of A to B, and is thus expressed, A : B.

If $A > B$, the fr^{n} expressing the ratio of A to B, *i. e.* $\frac{A}{B}$, will be an improper fr^{n} greater than 1;

If $A = B$, the fr^{n} will become $\frac{A}{A}$, or unity;

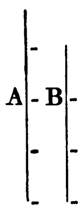
If $A < B$, the fr^{n} will become $\frac{A}{B}$, a proper fr^{n}.

Also, when the fr^{n} $\frac{A}{B}$ is greater than 1, the ratio of A to B is called a ratio of *greater* inequality, because $A > B$.

When the fr^{n} $\frac{A}{B} = \frac{A}{A} = 1$, the ratio is called a ratio of equality; and when the fr^{n} $\frac{A}{B}$ is a proper fr^{n}, the ratio is called a ratio of *less* inequality, because $A < B$.

67. If the num^{r} and den^{r} of a fr^{n} expressing ratio both consist of the same kind of *concrete* quantities, the ratio between these quantities will be an *abstract* number.

Ex. If the lines A and B represent 4 inches and 3 inches



respectively, then $A : B = \frac{A}{B} = \frac{4 \text{ in.}}{3 \text{ in.}} = \frac{4}{3}$; and if, instead of inches, these lines had represented 4 feet and 3 feet, or 4 lb. and 3 lb., the ratio of A to B would still be $\frac{4}{3}$, and of B to A would be $\frac{3}{4}$.

But if the num^{r} and den^{r} be of the same species of concrete quantity, but not expressed in the

same denomination, the ratio cannot be represented by an abstract quantity, until they both be reduced to the same denomination. Thus, $\frac{4 \text{ feet}}{9 \text{ inches}}$ does not $= \frac{4}{9}$ feet, or $\frac{4}{9}$ in. or $\frac{4}{9}$, but it $= \frac{48 \text{ in.}}{9 \text{ in.}} = \frac{48}{9} = \frac{16}{3}$, an abstract number.

If quantities be not of the same nature, there can be no ratio between them; thus $\frac{4 \text{ feet}}{5 \text{ minutes}}$ is no ratio at all, since we cannot compare the magnitude of 4 feet and of 5 minutes.

This article must be especially noticed, as upon it will depend the mode of working the Exs. which have been alluded to in (65).

68. We now see that any fraction, as $\frac{2}{5}$, has still a further meaning, beside what was stated in (21), viz. *the ratio of* 2 *to* 5, or, as it is written, 2 : 5; and $\frac{2}{5}$ may be called *a ratio*.

If, then, two other quantities be taken, as 4 and 10, and it is found that $\frac{4}{10} = \frac{2}{5}$, then we see that the ratio of 4 to 10 is equal to the ratio of 2 to 5, and this fact we express either thus,

$$4 : 10 = 2 : 5, \text{ or } 4 : 10 :: 2 : 5;$$

the latter expression is thus read—4 is to 10 as 2 is to 5.

When, therefore, two ratios, as $\frac{4}{10}$ and $\frac{2}{5}$, are placed before us, and we learn that they are equal, we say that the four quantities, 4, 10, 2, 5, are *Proportionals.*

69. We may now define Proportion to consist in the equality of two ratios. Of the above-mentioned numbers 4, 10, 2, 5—4 and 5 are called the *extremes;* 10 and 2 are called the *means,* because they are intermediate between the extremes.

Also, when 4 quantities are given, and we wish to ascertain whether they are proportionals or not, we must see if the fr^n expressing the ratio of the 1st and 2nd = the fr^n expressing the ratio of the 3rd and 4th. Thus, if I take the numbers 5, 6, 7, 8, and wish to try whether they are proportionals, I compare $\frac{5}{6}$, the ratio of the first pair, and $\frac{7}{8}$, the ratio of the second pair; and if these fr^{ns} are proved unequal, the above 4 quantities are not proportionals.

Now, it was shown in (50), that to compare two $frac^l$ quantities we must bring them to some Com. Den^r. Reducing the fr^{ns} $\frac{5}{6}$ and $\frac{7}{8}$ to a C. D. 48, we have to compare

$$\frac{5 \times 8}{48} \text{ and } \frac{6 \times 7}{48} \qquad \text{(Q)}$$

And since $\frac{40}{48}$ is not $= \frac{42}{48}$, therefore $\frac{5}{6}$ is not $= \frac{7}{8}$, and 5, 6, 7, 8, are not proportionals.

It will be seen in line (Q) that the test as to whether or not the four proposed quantities be proportionals, consists in multiplying, 1st, the two extremes 5, 8, and 2ndly, the two means 6, 7: and if these products be equal, the four quantities are proportionals; if they be not equal, the four quantities are not proportionals.

This operation may generally be performed *mentally*: thus, if I take 5, 8, 9, 16, I say $5 \times 16 = 80$, and $8 \times 9 = 72$; therefore these quantities are not proportionals. Again, if I take 4, 9, 5, $11\frac{1}{4}$, since the product of the extremes 4 times $11\frac{1}{4} = 4 \times \frac{45}{4} = 45$, and the product of the means $9 \times 5 = 45$; therefore these last four quantities are proportionals. (See Appendix, Art. Ratio.)

70. Having now explained how to tell when four quantities are proportionals, we proceed to show how, when *three* numbers are given, we may find a fourth

number, such that the other three and this fourth shall be proportionals. This is sometimes called finding a fourth proportional to three given terms. But, in order to explain this process the more readily, I will first explain some operations in Fractions which have not yet been noticed.

71. We have seen (28, 29) that to multiply a fr^{n} by any number, we may either multiply the num^{r} or divide the den^{r} by this multiplier. Thus, if I wish to multiply $\frac{4}{10}$ by 10, it becomes

$$\frac{4\times 10}{10}=\frac{4\times \cancel{10}}{\cancel{10}}=4;$$

$$\text{or, } \frac{4}{10\div 10}=\frac{4}{1}=4:$$

therefore, if I wish to multiply a fr^{n} by its own den^{r}, I obtain the correct result by merely taking away that den^{r}.

Now, if I multiply two equal quantities by the same multiplier, the products will be equal. If, therefore, in the equation,

$$\frac{4}{10}=\frac{2}{5} \qquad (\text{I.})$$

I multiply both sides by 10; the first side becomes 4, as just shown, and the second side becomes

$$\frac{2\times 10}{5}; \text{ therefore } 4=\frac{2\times 10}{5} \qquad (\text{II.})$$

By comparing (I.) and (II.) I find that I have transferred a den^{r} of one side of the equation into the num^{r} of the other, without disturbing the equality.

Again, it has been shown (9) that if I have a product, as $7\times 5\times 3$, and I wish to divide this product by one or more of its factors, I have merely to strike out such one or more factors; *i. e.* if I wish to divide $7\times 5\times 3$ by 5, I

remove the 5, and have as quotient 7×3; therefore, if in equation (II.) I wish to divide the right-hand side by 2, I have only to cancel the 2, and leave as the quotient $\frac{10}{5}$: also, dividing the left-hand side by 2, by making 2 as the den^r (22), but *not performing* the division, I have the quotient $\frac{4}{2}$.

Since, then, I have divided both sides of the equation (II.) by 2, the quotients must be equal;

$$i.\,e.\ \frac{4}{2} = \frac{10}{5} \qquad \text{(III.)}$$

On comparing (I.) and (III.) it will be seen that I have now transferred a factor 2 from the num^r of the right side to the den^r of the left; or made a multiplier of one side into a divisor of the other: also that I have transferred a quantity 10 from the den^r of the left side to the num^r of the right side. And these processes might have been performed, whatever had been the two fractions given in (I.)

72. We may now, therefore, conclude, that if two equal fractions are placed before us, we can transfer any factor from the num^r of one side to the den^r of the other, and from the den^r of one side to the num^r of the other, and the equality will still exist. But it must be distinctly noticed, that we do not change the places of these factors by subtraction and addition, but by division and multiplication: thus, if I have

$$\frac{3}{5} = \frac{6}{10} \text{ or } = \frac{2 \times 3}{10} \qquad \text{(IV.)}$$

and I wish to remove a 2 from the right-hand num^r to the left-hand den^r, I *divide* the num^r by the 2, and *multiply* the other den^r by it, and I have

$$\frac{3}{5 \times 2} = \frac{1 \times 3}{10}:$$

now, it is not necessary to write this fig. 1 as I have done, because $1 \times 3 = 3$; but let us remove the 3 from the right-hand num^r, and I have

$$\frac{3}{5 \times 2 \times 3} = \frac{1 \times 1}{10} = \frac{1}{10}.$$

It appears, then, that if I remove *all* the factors, *i. e.* the whole num^r, I must remember that I leave as many quotients 1 as I removed factors, and therefore the num^r or den^r from which the factors have been taken = the product of all these figures 1, and is therefore = 1. If this product, which = 1, occurs in the den^r, we do not write it down, because $\frac{6}{1} = 6$.

I will illustrate by an example this power of changing the places of num^r and den^r, as just explained.

$$\text{We know that } \frac{3}{7} = \frac{15}{35}; \qquad \text{(I.)}$$

Change, first the 7, and secondly the 15, and we have

$$3 = \frac{7 \times 15}{35} \quad \text{(II.)} \quad \text{and} \quad \frac{3}{15} = \frac{7}{35} \quad \text{(III.)}$$

Next, in III., change first the 3, and secondly the 35, and we have

$$\frac{1}{15} = \frac{7}{3 \times 35} \quad \text{(IV.)} \quad \text{and} \quad \frac{35}{15} = \frac{7}{3} \quad \text{(V.)}$$

Comparing (I.) and (V.), I find that (V.) is (I.) inverted: hence, I learn that if two fr^ns are equal, they will be equal when inverted. However complicated the two equal fractions may be, this will make no difference:

$$\text{Thus, if } \frac{3\frac{1}{2}}{7\frac{1}{5}} = \frac{35}{72},$$

we can say, $3\frac{1}{2} = \frac{7\frac{1}{5} \times 35}{72}$, or $72 = \frac{7\frac{1}{5} \times 35}{3\frac{1}{2}}$, or $\frac{72}{35} = \frac{7\frac{1}{5}}{3\frac{1}{2}}$.

73. Having established the truth of the above operations, I proceed to solve the following question.

If three numbers be given, as 5, 6, 10, what must be taken as a fourth number, such that the four, when taken in order, shall be proportionals?

Now, as I do not know (or, at least, the learner does not know,) the required number, let the letter N stand for or represent this number; and I have now to try and find what this N must be.

Since, then, 5, 6, 10, and N are required to be proportionals, therefore we must have the ratio between the first pair 5, 6, = that between the second pair 10, N; *i. e.*

$$\frac{5}{6} = \frac{10}{N}; \qquad (R)$$

change 5 and N, as shown above, and we have $\frac{N}{6} = \frac{10}{5}$.

Next, throw the 6 into the right-hand num^r, and we find

$$N = \frac{6 \times 10}{5} \qquad (S)$$

$$= \frac{6 \times \cancel{10}^{2}}{\cancel{5}} = 12.$$

Put 12 for N in (R), and we have $\frac{5}{6} = \frac{10}{12}$, which we know to be true, and therefore 5, 6, 10, 12, are proportionals.

The result of this article must be most carefully noticed; for in observing (S) we learn that the required 4th number N was thus formed from the three former ones, 5, 6, 10: viz. that it is the value of this fr^n—the product of the 2nd and 3rd terms, divided by the first; or it $= \frac{\text{2nd} \times \text{3rd}}{\text{1st}}$.

Exs. 15. Find a fourth proportional to each of the following sets of numbers:—

I. 3, 7, 15; 5, 2, 11; 9, 8, $\frac{1}{7}$

II. $3\frac{1}{2}$, $7\frac{3}{4}$, $\frac{1}{5}$; $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$; $\frac{1}{9}$, $8\frac{1}{2}$, $\frac{1}{10}$

III. 19, 18, 17; 17, 18, 19.

74. We have shown that, in order that there may exist

a ratio between two quantities, they must be of the same kind; since, then, there is to be a proportion among the three given quantities and the fourth, therefore between the 1st and 2nd there must be a ratio, and an equal ratio between the 3rd and 4th. Hence we see that the second pair must be of the same kind; that is, the 4th quantity, when found, ought to be of the same kind as the 3rd. Now, if we observe the fr^{n} $\frac{\text{2nd} \times \text{3rd}}{\text{1st}}$ in the last article, we notice, that since it is allowed that the 2nd and 1st are alike in kind, therefore the $\frac{\text{2nd}}{\text{1st}}$ will be an abstract number (67); and therefore the 4th, which $= \frac{\text{2nd}}{\text{1st}} \times$ 3rd, $=$ 3rd multiplied by some abstract number; *i. e.* it will be the 3rd term repeated as many times as there are units in this number, and therefore will be of the same kind as that in which the third term was expressed.

Also, if the 2nd term be greater than the 1st, $\frac{\text{2nd}}{\text{1st}}$ will be a fr^{n} > 1, and this multiplier of the 3rd term will be greater than 1, and the 4th term greater than the 3rd; but if the 2nd term be less than the 1st, then $\frac{\text{2nd}}{\text{1st}}$ will be a proper fr^{n}, and the multiplier less than 1, and the 4th term less than the 3rd. (See Appendix, Art., *Proportion.*)

75. We may here notice a form of expression that we familiarly use, when we say, that two quantities are proportional to one another; as, for instance, that the amount of a servant's wages is proportional to the length of his service. Here, apparently, two quantities of different kinds are compared, viz. money and time; but, in reality, four quantities are implied, viz. two periods of time, and two

amounts of wages. For since one year and one year's wages may be taken as fixed standards, by which we may measure other periods and other amounts; therefore I mentally compare any proposed length of service and the corresponding amount of wages with these fixed standards; and the original expression means this—that there is the same ratio between any given length of service and one year, that there is between the amount of wages for that service and one year's wages; or, that these two periods of service and two amounts of wages together form a proportion in this order,

Given time : 1 year :: wages for that time : wages for 1 year.

If we wish to ascertain whether two quantities are what is here called proportional to one another, according to the common usage of the words, we may try if doubling the one quantity causes the other to be doubled; as for example, in this instance—will the wages be doubled, if the time of service be doubled? Since this is the case, therefore the wages and time are proportional in the sense explained above.

The correct, though not common mode of expressing this proportion, is to say, that the amount of wages *varies as* the length of the time of service.

76. But sometimes this proportion occurs in a different form; as, when a man has a certain number of miles to walk in any time; then, if he quickens his rate of going, he can complete his task in less time. In this case it would be incorrect to say that the length of time was in proportion to the rate of walking; for, according to the test given above, I ask—if the rate be doubled, will the number of days be *doubled?* the answer is, no: on the contrary, it

will be *halved*. There is here, then, evidently a proportion (taking the same word in the common usage,) but of a different kind, viz., that as one number is increased by multiplication, the other is correspondingly diminished by division.

As, in the first instance, it was said that the wages *varied as* the time, so, in this second example, the relation between the two quantities is correctly expressed by saying, that the number of days *varies inversely as* the rate of walking; or, as it is commonly said—in the first Ex., the wages were *directly* proportional to the time; and in the second Ex. the number of days is *inversely* proportional to the rate.

Proportion has here been shewn to involve 4 quantities; these need not, however, be all different. Thus, if I find 3 quantities, whereof the ratio between the 1st and 2nd = that between the 2nd and 3rd, then the three quantities are said to be proportional: thus, 3, 6, 12 are proportionals, because $\frac{3}{6} = \frac{6}{12}$, or $3 : 6 = 6 : 12$. (T)

Also, to find this 3rd propor[l] 12, when the two former ones are given, we must compare line (T) with an ordinary proportion containing 4 quan[s], and we shall see that 12, the required 3rd, is in the 4th place, and the 2nd quan[y], 6, is repeated in the 3rd place: hence, the usual equation 4th = $\frac{\text{2nd} \times \text{3rd}}{\text{1st}}$, will become 3rd = $\frac{\text{2nd} \times \text{2nd}}{\text{1st}}$; and, if 3 and 6 be the 1st and 2nd terms, we have 3rd = $\frac{6 \times 6}{3} = 12$.

Exs. 16.

Find a third proportional to each of the following pairs of numbers:—

IV. 8, 12; 12, 18; 15, 16; 20, 15;

V. $\frac{1}{3}$, $\frac{1}{4}$; $2\frac{1}{3}$, $\frac{3}{5}$; $\frac{1}{2}$ of $\frac{3}{8}$, $\frac{2}{7}$ of 3.

REDUCTION OF FRACTIONS.

(Thrower, Case XIII.)

77. We are now able to perform the operation alluded to in (65) viz., to express one quantity in terms of another, or as a frac^{l} part of another. This fr^{n} connecting the two quantities will shew the ratio between them, and may be either proper or improper.

For example—to express 9d. in terms of £1, is to see what ratio 9d. bears to £1. Hence, according to what has been explained concerning ratio, we say,

$$9\text{d. in terms of } £1 = 9\text{d.} : £1.$$

$$= \frac{9\text{d.}}{£1} = \frac{9\text{d.}}{240\text{d.}} = \frac{9}{240} = \frac{3}{80};$$

and transferring the £1 from the den^{r} of the left-hand fr^{n} to the num^{r} of the right-hand fr^{n} by (71), I have 9d. $= \frac{3}{80} \times £1$, or $\frac{3}{80}$ of £1. This result may also be obtained as follows:—

I have to express pence in terms of £1.

Now, 240 pence = £1; therefore, taking $\frac{1}{240}$th part of both sides—

$$1 \text{ penny} = \frac{1}{240} \text{ of } £1;$$

and multiplying by 9,

$$9 \text{ pence} = \frac{9}{240} \text{ of } £1$$

$$= \frac{3}{80} \text{ of } £1;$$

and transferring the £1 from the right-hand numerator to the left-hand denominator, we have, as before,

$$\frac{9 \text{ pence}}{£1} = \frac{3}{80}.$$

It will be observed, that by this second method we have an independent proof, that the ratio between two concrete numbers of like kind, as pence, pounds, &c., is an abstract number.

78. Either of the two methods just shown may be employed; but I prefer the former, because it more decidedly keeps the idea of ratio before the mind; and an Ex. so worked can be written out in a more condensed form than by the latter method. It will be noticed that the two concrete quantities are *both* to be reduced to some com^{n} den^{n}, in order that the ratio between them may be expressed: thus, in the last Ex., both were reduced to pence.

Ex. II. Express 7s. 7½d. in terms of 10s. 6d.

Here I exhibit both numerator and denominator in terms of pence: and write

$$\frac{7s.\ 7\frac{1}{2}d.}{10s.\ 6d.}=\frac{91\frac{1}{2}d.}{126d.}=\frac{\frac{183}{2}}{126}=\frac{\overset{61}{\cancel{183}}}{2\times\underset{42}{\cancel{126}}}=\frac{61}{84}.$$

I might have reduced both numerator and denominator to half-pence, but a pupil would not easily reduce 7s. 7½d. to half-pence mentally; but he might write it in pence and a fraction of a penny, as I have done, and then the successive steps of the work will be seen more clearly.

Referring to Ex. I., we read—$\frac{9d.}{£1.}=\frac{3}{80}$; and since, by (72), two equal fractions may be inverted, without disturbing the equality; therefore $\frac{£1}{9d.}=\frac{80}{3}$; or, changing the 9d. to the right-hand numerator,

$$£1=\frac{80}{3}\times 9d.;\ \text{or}=\frac{80}{3}\text{ of }9d.$$

Hence we see that in every Ex. where we have to express one quantity in terms of a second, we can express this second in terms of the first, by merely inverting the ratio which connected the first and second. Thus, in Ex. II.,

$$\text{since } 7s.\ 7\tfrac{1}{2}d.=\frac{61}{84}\text{ of }10s.\ 6d.;$$

$$\text{therefore } 10s.\ 6d.=\frac{84}{61}\text{ of }7s.\ 7\tfrac{1}{2}d.$$

79. Sometimes fractions may be involved in both num^{r} and den^{r}, as in

Ex. III. Express 1¾ of 2a. 1r. in terms of 3 acres 2½ roods.

Here, bringing both quantities into roods, and changing *of* into $\times$, we have

$$\frac{1\frac{3}{4}\text{ of }2\text{a. }1\text{r.}}{3\text{a. }2\frac{1}{2}\text{r.}}=\frac{\frac{7}{4}\times 9\text{r.}}{14\frac{1}{2}\text{r.}}=\frac{\frac{7}{4}\times 9}{\frac{29}{2}}$$

$$=\frac{7}{\cancel{4}_{2}}\times\frac{9}{1}\times\frac{\cancel{2}}{29}=\frac{63}{58}.$$

This should be left as an improper fraction, because I can then read off this desired result—that

$$1\tfrac{3}{4}\text{ of }2\text{a. }1\text{r.} : 3\text{a. }2\tfrac{1}{2}\text{r.} = 63 : 58;$$

or that the ratio of these two portions of land is that of 63 to 58.

As in (64) it was shown to be advisable to be able to express aliquot parts of any den^{n} in terms of lower den^{ns}, so, with a little experience, a pupil will, by a reverse process, mentally work easy Exs. in this Case, so as at once to see, that 5s. in terms of £1 $=\frac{1}{4}$; that 6s. 8d. : £1 $=\frac{1}{3}$, and so on: and a readiness in performing such simple reductions will often materially shorten the labour of more complicated Exs. Thus, to express $\frac{7}{8}$ of 13s. 4d. in terms of 10s.:—

By observing that 13s. 4d. $=\frac{2}{3}$£, and 10s. $=\frac{1}{2}$£, I have

$$\frac{\frac{7}{8}\text{ of }13\text{s. }4\text{d.}}{10\text{s.}}=\frac{\frac{7}{8}\text{ of }\frac{2}{3}£}{\frac{1}{2}£}=\frac{7}{\cancel{8}_{\cancel{4}_{2}}}\times\frac{\cancel{2}}{3}\times\frac{\cancel{2}}{1}=\frac{7}{6};$$

or, $\frac{7}{8}$ of 13s. 4d. $=\frac{7}{6}$ of 10s.

By the ordinary method I should have reduced both 13s. 4d. and 10s. to pence or fourpences, and the resulting fraction would have required much heavier reduction than mine has needed.

80. When either the num^{r} or den^{r} of the left-hand fr^{n} requires much reduction to a lower den^{n}, it is better to express by signs the multiplication requisite to perform this reduction, rather than to perform the mult^{n}, and put down the result; because, when either num^{r} or den^{r} is thus expressed in factors, it is in the best form for detecting the probability of any cancelling taking place, so as to present the resulting fr^{n} in its lowest terms.

For example, in reducing 5 drs. 1 sc. 15 grs. to the fraction of 1 lb., I must bring the 1 lb. to grains, and I write

$$\frac{5\text{ dr. }1\text{ sc. }15\text{ grs.}}{1\text{ lb.}} = \frac{16\text{ sc. }15\text{ grs.}}{1\times12\times8\times3\text{ sc.}} = \frac{335\text{ grs.}}{12\times8\times3\times20\text{ grs.}} \qquad (U)$$

$$= \frac{\overset{67}{\cancel{335}}}{12\times8\times3\times\underset{4}{\cancel{20}}} = \frac{67}{96\times12} = \frac{67}{1152}.$$

But in this Ex. we may also notice that the last denomination 15 grs. may be very readily expressed as a fraction of the preceding denomination, viz. scruples, for 15 grs. $= \frac{15}{20}$ sc. $= \frac{3}{4}$ sc.; therefore instead of the two latter fractions in line (U), I should write

$$= \frac{16\tfrac{3}{4}\text{ sc.}}{12\times8\times3\text{sc.}} = \frac{\frac{67}{4}}{12\times8\times3} = \frac{67}{4\times12\times8\times3} = \frac{67}{12\times96}$$

$$= \frac{67}{1152}.$$

Of these five fractions the last four are merely reductions to a simpler form; and the number of steps which a pupil may have to write down in working Exs. similar to the one above will depend, partly on the magnitude of the numbers involved, and partly on his own quickness in working mentally.

I will give one more Ex. in which the work is condensed, but still the successive operations are intelligible.

Express $\frac{11}{12}$ of $4\frac{1}{4}$d. in terms of 9s. $7\frac{1}{2}$d.

$$\frac{\frac{11}{12}\text{ of }4\tfrac{1}{4}\text{d.}}{9\text{s. }7\tfrac{1}{2}\text{d.}} = \frac{\frac{11}{12}\times\frac{17}{4}\text{d.}}{115\tfrac{1}{2}\text{d.}} = \frac{\cancel{11}}{12}\times\frac{17}{\underset{2}{\cancel{4}}}\times\frac{\cancel{2}}{\underset{21}{\cancel{231}}} = \frac{17}{504};$$

$$\text{or, } \frac{11}{12}\text{ of }4\tfrac{1}{4}\text{d.} = \frac{17}{504}\text{ of 9s. }7\tfrac{1}{2}\text{d.}$$

Exs. 17.

1.	Express	1s. 3d.,	and 17s. 6d....	in terms	of £1.
2.	„	17s. $7\frac{1}{2}$d.,	„ 25s. $9\frac{1}{4}$d. ..	„	1 guinea.
3.	„	12s. $7\frac{5}{7}$d.,	„ 2s. $11\frac{1}{3}$s. ..	„	15s. 6d.
4.	„	$\frac{5}{12}$ of a moidore,	and $3\frac{3}{4}$ groats	„	$12\frac{1}{2}$ guineas.
5.	„	$\frac{3}{7}$ £,	and $\frac{1}{1\frac{3}{8}}$ of 7s........	„	1 farthing.
6.	„	3 qrs. $1\frac{1}{3}$ nls.,	and $\frac{1}{2}$ in. ..	„	1 Flem. ell.

7.	„	3 oz. 5 drs. $1\frac{1}{4}$ sc.	in terms of	1 lb.
8.	„	$\frac{5}{7}$ of $1\frac{1}{8}$ guineas	„	£5.
9.	„	$\frac{3}{5}$ of $1\frac{5}{8}$ of an acre	„	1 sq. yd.
10.	„	$\frac{10}{11}$ of $1\frac{4}{8}$ of 3 miles......	„	5 miles.
11.	„	1 hour	„	$3\frac{1}{4}$ of $365\frac{1}{4}$ days.
12.	„	$2\frac{7}{10}$ of $\frac{3}{4}$ of 15 solid inches	„	$3\frac{3}{8}$ cubic yds.

SIMPLIFICATION OF FRACTIONS; INCLUDING EXAMPLES IN THE COMPOUND RULES, WHICH INVOLVE FRACTIONS.

(**Thrower, Case XIV.**)

81. Under this head are arranged—1st. Exs. involving the expressing of frac[l] quan[s] in positive terms, and requiring both Addition and Subtraction, as—

I. $\frac{5}{8}$£ $+\frac{3}{4}$ of 21s.$-\frac{7}{8}$ of 27s.

2ndly, Exs. in Compound Addition, Subtraction, Multiplication, and Division, in which fractional qnantities are involved. I give one of each.

II. £3 6s. $8\frac{5}{8}$d. + £2 11s. $8\frac{7}{8}$d. + £4 13s. $9\frac{3}{8}$d.

III. £7 16s. $8\frac{5}{8}$d.$-$£3 11s. $10\frac{7}{8}$d.

IV. £8 14s. $2\frac{7}{12}$d. $\times$ $46\frac{5}{8}$d.

V. £38 3s. $4\frac{3}{8}$d. $\div$ $13\frac{7}{8}$d.

82. In Ex. I. we may use either of the following methods, viz. reduce all the quantities to the same denomination, and then find the value of their sum as in (62); or express in positive terms each fraction separately, and then find the value of the whole by Compound Addition and Subtraction.

By the first method, reducing the quantities to the fraction of £1, I have

$$\frac{5}{8}\text{£} + \frac{3}{4}\text{ of 21s.} - \frac{7}{8}\text{ of 27s.}$$

$$= \frac{5}{8}\text{£} + \frac{3}{4} \times \frac{21}{20}\text{£} - \frac{7}{8} \times \frac{27}{20}\text{£}$$

$$= \left(\frac{5}{8} + \frac{63}{80} - \frac{189}{160}\right)£$$

$$= \frac{100 + 126 - 189}{160}£ \qquad \text{(since L. C. D.} = 160\text{)}$$

$$= \frac{226 - 189}{160}£ = \frac{37}{160}£$$

and $\frac{37}{160}£ = \frac{37 \times \cancel{20}}{\underset{8}{\cancel{160}}}\text{sh.} = \frac{37}{8}\text{sh.} = 4\tfrac{5}{8}\text{sh.} = 4\text{s. } 7\tfrac{1}{2}\text{d.}$

By the second method—

$\frac{5}{8}£\ (64) = 12\text{s. } 6\text{d.}$

also, $\frac{3}{4}$ of $21\text{sh.} = \frac{63}{4}\text{sh.} = 15\tfrac{3}{4}\text{sh.} = 15\text{s. } 9\text{d.}$

and $\frac{7}{8}$ of $27\text{sh.} = \frac{189}{8}\text{sh.} = 23\tfrac{5}{8}\text{sh.} = £1\ 3\text{s. } 7\tfrac{1}{2}\text{d.}$

therefore $\frac{5}{8}£ + \frac{3}{4}$ of $21\text{s.} - \frac{7}{8}$ of $27\text{s.} = 12\text{s. } 6\text{d.} + 15\text{s. } 9\text{d.} - £1\ 3\text{s. } 7\tfrac{1}{2}\text{d.}$

$$= £1\ 8\text{s. } 3\text{d.} - £1\ 3\text{s. } 7\tfrac{1}{2}\text{d.}$$

$$= 4\text{s. } 7\tfrac{1}{2}\text{d.}$$

83. Ex. II. $£3\ 6\text{s. } 8\tfrac{5}{6}\text{d.} + £2\ 11\text{s. } 8\tfrac{7}{8}\text{d.} + £4\ 13\text{s. } 9\tfrac{2}{3}\text{d.}$

Commencing with the addition of the fractional parts, I have

$$\frac{5}{6}\text{d.} + \frac{7}{8}\text{d.} + \frac{2}{3}\text{d.} = \frac{20 + 21 + 16}{24}\text{d.}$$

$$= \frac{57}{24}\text{d.} = \frac{19}{8}\text{d.} = 2\tfrac{3}{8}\text{d.}$$

and carrying the 2d. to the row of pence, I complete the sum as in Compound Addition; and the whole amount = $£10\ 12\text{s. } 3\tfrac{3}{8}\text{d.}$

84. Ex. III. $£7\ 16\text{s. } 8\tfrac{5}{6}\text{d.} - £3\ 11\text{s. } 10\tfrac{7}{8}\text{d.}$

Commencing with the subtraction of the fractional parts, I observe that $\frac{5}{6} < \frac{7}{8}$; therefore, borrowing 1d. from the 8d. in the former quantity, I have

$$1\tfrac{5}{6}\text{d.} - \tfrac{7}{8}\text{d.} = \left(\frac{11}{6} - \frac{7}{8}\right)\text{d.} = \frac{44 - 21}{24}\text{d.} = \frac{23}{24}\text{d.}$$

and completing the sum by Comp[d] Sub[n], the whole difference is $£4\ 4\text{s. } 9\tfrac{23}{24}\text{d.}$

Ex. IV. £8 14s. $2\frac{7}{12}$d. × $46\frac{8}{9}$d.

Now, since to multiply by $46\frac{8}{9}$ is to repeat the multiplicand 46 times and $\frac{8}{9}$ times, therefore, if I multiply the given comp^d quan^y by 46, and then take $\frac{8}{9}$ of the same, the sum of the two products will be the product required, viz. $46\frac{8}{9}$ times £8 14s. $2\frac{7}{12}$d. Proceeding as in Comp^d Mult^n, and working in the margin those parts of the mult^n which involve fractions, I have

£8	14s.	$2\frac{7}{12}$d.	
		$9 \times 5 + 1 = 46$	
£78	7s.	$11\frac{1}{4}$	
		5	
£391	19s.	$8\frac{1}{4}$	= 45 times
£8	14s.	$2\frac{7}{12}$	= 1 „
£7	14s.	$10\frac{8}{27}$	= $\frac{8}{9}$ „
£408	8s.	$9\frac{7}{54}$	= $46\frac{8}{9}$ „

$$\left[\overset{3}{\cancel{9}} \times \frac{7}{\underset{4}{\cancel{12}}}\text{d.} = \frac{21}{4}\text{d.} = 5\tfrac{1}{4}\text{d.}\right]$$

And $\frac{8}{9}$ of (£8 14s. $2\frac{7}{12}$d.)

$$= \frac{8 \times (\text{£}8\ 14\text{s.}\ 2\frac{7}{12})}{9}$$

$$= \frac{\text{£}69\ 13\text{s.}\ 8\frac{2}{3}\text{d.}}{9}$$

$$\left[\text{for } \overset{2}{\cancel{8}} \times \frac{7}{\underset{3}{\cancel{12}}}\text{d.} = \frac{14}{3}\text{d.} = 4\tfrac{2}{3}\text{d.}\right]$$

= £7 14s. $10\frac{8}{27}$d.

The $\frac{8}{27}$d. is thus obtained: after dividing the pence by 9, there remained 2d.; therefore I had to divide $2\frac{2}{3}$d. by 9—the quotient

$$= \frac{2\frac{2}{3}\text{d.}}{9} = \frac{\frac{8}{3}}{9}\text{d.} = \frac{8}{27}\text{d.}$$

$$\text{Sum of the fractions} = \left(\frac{1}{4} + \frac{7}{12} + \frac{8}{27}\right)\text{d.} = \frac{27 + 63 + 32}{108}\text{d.}$$

$$= \frac{122}{108}\text{d.} = \frac{61}{54}\text{d.}$$

$$= 1\tfrac{7}{54}\text{d.}$$

Again, $46\frac{8}{9} = \frac{422}{9}$; therefore I may multiply the given sum by $\frac{422}{9}$; *i. e.* multiply by 422, and divide by 9. But as this second method involves more labour, and would therefore

be not so generally used, it is not worth while to work the Ex. out, especially as the process of Compound Multiplication, where fractions are involved, is sufficiently shown in the former method.

85. Ex. V. £38 3s. $4\frac{3}{5}$d. $\div$ $13\frac{7}{9}$.

There is but one mode of working this Ex., viz. to reduce the divisor to an imp^r fr^n—invert it, and proceed as in multiplication.

Now, $13\frac{7}{9} = \frac{124}{9}$; and this, when inverted, becomes $\frac{9}{124}$. Therefore the Ex. becomes £38 3s. $4\frac{3}{5}$d. $\times \frac{9}{124}$; *i. e.* I have to multiply the given sum by 9, and divide by 124. The work will be as follows:—

$$
\begin{array}{rrrrl}
 & \pounds & s. & d. & \\
 & 38 & 3 & 4\frac{3}{5} & \\
 & & & 9 & \qquad 9 \times \frac{3}{5}\text{d.} = \frac{27}{5}\text{d.} = 5\frac{2}{5}\text{d.} \\
\hline
124) & 343 & 10 & 5\frac{2}{5} & (\pounds 2 \\
 & \underline{248} & & & \\
 & 95 & & & \\
 & \underline{20} & & & \\
124) & 1910 & & & (15s. \\
 & \underline{124} & & & \\
 & 670 & & & \\
 & \underline{620} & & & \\
 & 50 & & & \\
 & \underline{12} & & & \\
124) & 605 & & & (4d. \\
 & \underline{496} & & & \\
 & 109 & & &
\end{array}
$$

There now remain 109d. and $\frac{2}{5}$d., which have not yet been divided by 124; the quotient =

$$\frac{109\frac{2}{5}\text{d.}}{124} = \frac{\frac{547}{5}}{124}\text{d.} = \frac{547}{5 \times 124}\text{d.} = \frac{547}{620}\text{d.}$$

and the whole quotient is £2 15s. $4\frac{547}{620}$d.

Exs. 18. Find the value of

1. $\frac{3}{4}$ guinea + $\frac{3}{14}$ of 13s. 4d. − $\frac{1}{7}$ of 27s.
2. $\frac{2}{3}$ ton + $\frac{7}{8}$ lb. + $\frac{5}{12}$ cwt. (in lbs. and oz.)

3. $\frac{3}{5}$ mile $-$ $\frac{4}{9}$ furlong $+$ $\frac{1}{11}$ pole.
4. $1\frac{1}{3}$ acr. $+$ $\frac{4}{7}$ rood $-$ $\frac{3}{5}$ sq. yd.
5. £7 16s. $8\frac{1}{3}$d. $+$ £138 4s. $10\frac{5}{8}$d. $+$ £78 6s. $3\frac{5}{8}$d.
6. 27 yds. 1 ft. $8\frac{2}{3}$ in. $+$ 5 yds. 0 ft. $9\frac{2}{3}$ in. $-$ 2 ft. $0\frac{5}{8}$ in.
7. 4 tons 3 cwts. $2\frac{5}{9}$ qrs. $+$ 15 cwt. $57\frac{1}{2}$ lbs. $-$ 3 qrs. 13 lbs. $5\frac{1}{3}$ oz.
8. £463 12s. $4\frac{3}{5}$d. $\sim$ £764 19s. $3\frac{1}{7}$d.
9. £114 8s. $0\frac{6}{7}$d. $\times$ $8\frac{5}{9}$.
10. £15 9s. $8\frac{1}{13}$d. $\times$ $23\frac{6}{7}$.
11. 17 yrs. 3 mo. 2 wks. 3 dys. $\times$ $375\frac{8}{11}$.
12. £135 0s. $11\frac{3}{16}$d. $\div$ $12\frac{1}{12}$.
13. 1684 lbs. 10 oz. 17 dwt. $11\frac{1}{3}$ grs. $\div$ $47\frac{2}{3}$.
14. £13 15s. $9\frac{3}{11}$d. $\times$ $17\frac{2}{3}$.
15. (£3 8s. $7\frac{1}{4}$d. $+$ £2 17s. $5\frac{1}{6}$d.) $\div$ $15\frac{1}{3}$.

MISCELLANEOUS EXAMPLES.

86. These, of course, consist of questions involving one or more of the various processes which we have considered; and all that can be done towards guiding the pupil in working such Exs. is, merely to assist him in judging what operations are required in solving any proposed question.

What was said in (60) applies more particularly here, viz., that it is necessary most carefully to note the *signs*, or rather the words which express signs. As an instance of this caution, I will consider the subjoined Ex.; and in order to work it, I shall endeavour to see what operations are intended by the words employed in the question.

Ex. I. Multiply the sum of $\frac{2}{3}$, $\frac{3}{4}$ of $\frac{5}{6}$, and 4, by $7\frac{1}{12}$.

Now, *the sum of* means addition, or $(+)$, and *of* means $(\times)$; therefore, this Ex. expressed in signs is $\left\{\frac{2}{3}+\left(\frac{3}{4}\times\frac{5}{6}\right)+4\right\}\times 7\frac{1}{12}$. This is now a mere Ex. of simplification as in (60); and the remaining work may be completed by a pupil.

87. Ex. II. How many persons may receive each $3\frac{5}{6}$s. out of £$13\frac{1}{7}$?

This is only a Reduction sum, involving fractions; and, in plainer language, means—how often are $3\frac{5}{6}$s. contained in £$13\frac{4}{5}$? or, if I divide £$13\frac{4}{5}$ by $3\frac{5}{6}$s., what is the quotient?

The quotient required is $\dfrac{£13\frac{4}{5}}{3\frac{5}{6}\text{s.}}$

$$= \frac{13\frac{4}{5}\times 20\text{s.}}{3\frac{5}{6}\text{s.}} = \frac{\frac{69}{5}\times\frac{20\text{s.}}{1}}{\frac{23}{6}\text{s.}} \qquad \text{(W)}$$

$$= \frac{\overset{3}{\cancel{69}}}{\cancel{5}}\times\frac{\overset{4}{\cancel{20}}}{1}\times\frac{6}{\cancel{23}} = 72.$$

By (67) we know that the second fraction in line (W) will be an abstract number; and the result, 72, shows that $3\frac{5}{6}$s. are contained 72 times in £$13\frac{4}{5}$.

88. Ex. III. Compare $\frac{3}{7}$£, $\frac{3}{8}$ of a guinea, and $\frac{2}{3}$ of 11s. $10\frac{1}{4}$d.

We cannot compare quantities without bringing them to some common name; *i. e.* in order to compare fractional quantities, we must reduce them to their L. C. D.

These quantities may therefore either be expressed in positive terms, as pounds, shillings, pence, &c., and then compared; and this is the best method when we wish to know the *difference* in value between the quantities; or they may be all expressed as fractions of one common quantity—say, of £1, or of 1 guinea, and then reduced to L. C. D. I use the latter method, because then I see more clearly the *ratio* between the quantities.

Reducing to frns of £1, I have

$$\frac{3}{8}\text{ guinea} = \frac{3}{8}\times\frac{21}{20}£ = \frac{63}{160}£.$$

$$\text{Also } \frac{2}{3}\text{ of 11s. }10\tfrac{1}{4}\text{d.} = \frac{2}{3}\times\frac{142\frac{1}{4}}{240}£ = \frac{\cancel{2}}{3}\times\frac{569}{\underset{2}{\cancel{4}}\times 240}£ = \frac{569}{1440}£.$$

Hence the 3 given quans, when expressed in the same denn, are

$$\frac{3}{7}£, \qquad \frac{63}{160}£, \qquad \text{and } \frac{569}{1440}£.$$

or, reduced to L. C. D. $\frac{4320}{10080}$, $\frac{3969}{10080}$, $\frac{3983}{10080}$.

or in the ratio of 4320 : 3969 : 3983.

89. I will work in full the following Ex., because upon the method of working it depends the solution of many questions in Arithmetic and Algebra.

Ex. IV. If $\frac{1}{3}+\frac{1}{4}+\frac{1}{12}$ of a number amount to 36, what is the number?

$$\text{Now } \frac{1}{3}+\frac{1}{4}+\frac{1}{12}=\frac{4+3+1}{12}=\frac{8}{12}=\frac{2}{3}.$$

And, by the question, this sum $= 36$;

i. e. $\frac{2}{3}$ of the number $= 36$;

therefore, dividing both sides by 2,

$\frac{1}{3}$ of the number $= 18$;

and, multiplying by 3,

$\frac{3}{3}$ of the number, *i. e.* the whole number $= 3 \times 18 = 54$.

90. The following is an Ex. which is to be worked upon a principle similar to the last.

Ex. V. If A can do a piece of work in 3 hours, B in 5 hours, and C in 7 hours, in what time can they do it, all working together?

Now, A can do the work in 3 hours;

therefore A can do $\frac{1}{3}$ of the work in 1 hour;

so, B „ $\frac{1}{5}$ „ 1 hour;

and C „ $\frac{1}{7}$ „ 1 hour;

therefore the three, A, B, C, working together, can perform

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7} \text{ or, } \frac{35+21+15}{105} \text{ or } \frac{71}{105} \text{ in 1 hour;}$$

therefore, if they do $\frac{71}{105}$ in 1 hour,

they will do $\frac{1}{105}$ in $\frac{1}{71}$ hour,

and therefore $\frac{105}{105}$, or the whole in $\frac{105}{71}$ hours;

that is, in $1\frac{34}{71}$ hours.

By observing the fr^{n} $\frac{71}{105}$, expressing the amount done in 1 hour, and $\frac{105}{71}$, the number of hours required for the whole work, we find that they are the reverse of one another; and this we might have expected, for the quantity of work done in any given time bears an *inverse* ratio to the amount of time in which it is done (76).

Here I have been finding the time of doing the whole work. I give one more Ex., in which it is required to find how much of a given piece of work can be done in any fixed time.

Ex. VI. A cistern is filled by two spouts in 20 and 24 minutes respectively, and emptied by a tap in 30 minutes; what portion of it will be filled in 15 minutes, when they are all left open together, the influx and efflux being uniform?

Since the 1st tap would fill the whole in 20 min.
therefore it would fill $\frac{1}{20}$th in 1 min.
so also, the 2nd „ $\frac{1}{24}$th in 1 min.

therefore $\frac{1}{20}+\frac{1}{24}$ are poured in by both together in 1 minute; but $\frac{1}{30}$ is discharged in 1 minute by the third tap; therefore, subtracting the quantity discharged from that poured in, we have remaining in the cistern at the end of 1 minute, $\frac{1}{20}+\frac{1}{24}-\frac{1}{30}$;

$$\text{and this} = \frac{6+5-4}{120} = \frac{11-4}{120}$$

$$= \frac{7}{120} \text{ in 1 minute;}$$

therefore at the end of 15 minutes there remains $\cancel{15} \times \frac{7}{\cancel{120}_{8}}$ or $\frac{7}{8}$; hence in 15 minutes the cistern will be seven-eighths full.

Exs. 19. A.

1. Explain the terms *product, dividend, quotient.*
2. With 263 for quotient, 368,469 for dividend, and 6 for remainder, what is the divisor?
3. Find the difference between one million and ten, and ten thousand and nine.

4. Write an equation involving the signs of addition, subtraction, multiplication, and division.

5. If a man's yearly income be known, what rule must be used to find his daily income? Ex. If the income be £532 5s. 10d., what is that per day?

6. What is the smallest number that can be exactly divided by the nine digits?

7. If light travels 192,000 miles per second, and the sun's light reaches us in $8\frac{1}{4}$ minutes, how far is the sun from the earth?

8. If 360 degrees be passed over in $365\frac{1}{4}$ days, how much is that per day?

9. To how many persons may £4 13s. 6d. each be given, out of a sum of £79 9s. 6d.?

10. What sum of money will be required to distribute to 10 poor men, 15 women, and 20 children, the respective sums of 2s., 1s., and 6d.?

11. Out of an income of 500 guineas, it is desired to lay by £150; what must be saved and spent daily?

12. Compare the product and quotient of $2\frac{1}{3}$ and $1\frac{7}{8}$.

13. Give the rules for multiplying and dividing fractions by whole numbers, and illustrate the processes by examples.

14. From unity subtract $\frac{1}{2}$, add $\frac{1}{3}$, subtract $\frac{1}{4}$, and add $\frac{1}{5}$; find the ratio of the result to the fraction $\frac{4}{5}$ of $1\frac{3}{8}$.

15. The sum realized by a bankrupt's estate is £7848, being $\frac{3}{16}$ of his debts, find the amount of the debts, and the dividend paid.

B.

1. How many dollars, each 4s. 6d., are contained in 20 moidores?

2. How many times will a wheel, 7 ft. 4 in. in circumference, turn in $3\frac{3}{4}$ miles?

3. If a railway carriage move 42 feet per second, how many miles is that per hour?

4. Find the value of 1000 oz. of silver plate, at 13s. 9d. per oz.; and shew how many times more valuable the same weight would be in gold, at £13 8s. $1\frac{1}{2}$d. per oz.

5. A yard of Cambridge butter weighs 1 lb., what should be the length of one penny-worth, at $17\frac{1}{2}$d. per lb.?

6. What sum must be divided among 18 men, and 9 women, so that each man may have £1 3s. 6d., and each woman $\frac{2}{3}$ of that sum?

7. A cask is required which can be filled by any one of the following measures, $\frac{1}{2}$ pint, $\frac{1}{2}$ gallon, 3 gals., 5 gals., and 9 gals.; find the smallest cask for this purpose.

8. Find the value of $\frac{7}{12} \sim \frac{9}{10}$ of a lb. Troy.

9. Out of a sum of £$18\frac{9}{10}$, how many persons may receive 2s. $7\frac{1}{2}$d. each?

10. Shew whether 3 : 19 is greater or less than $2\frac{1}{4} : 14\frac{1}{2}$; and find the change in the last term, that the four may form a proportion.

11. Of a package of cloth, $\frac{1}{3}$ is sold at 2s. 6d. a yard; $\frac{1}{4}$ at 1s. 6d. a yard, and the remaining 25 yds. at 1s. 10d., gaining 2d. per yd.; find the whole gain on the package.

12. Find the product and quotient of $\frac{2}{1\frac{3}{4}}$ and $\frac{1\frac{3}{8}}{4}$.

13. What fraction of £2 15s. is £2 14s. 9d.?

14. If $\frac{1}{8}$ of a ship be worth £73 1s. 3d., what part of her is worth £250 10s.?

15. Find the G. C. M. of 324 and 720, explaining the truth of the process.

C.

1. Express in words (1) the sum, and (2) the difference of 4,000,309 and 70,002.

2. Find in lbs. the value of 3 tons 14 cwt. 57 lbs.—1 ton 17 cwt. 3 qrs.

3. In coining 40,000 penny pieces, each costing $\frac{7}{8}$d., how much profit was made?

4. What number of steps 2 ft. 11 in. long, will be taken in walking 45 yards?

5. The cylinder of a railway engine is 18 in. long; how far will the piston travel in 500 revolutions of the driving wheel?

6. In 56 guineas, as many pounds, moidores, and half-crowns, how many groats?

7. If 25 yds. of cloth cost £7 17s. 6d., shew without using any proportion statement what is the value of 36 yds.

8. If £1 sterling be worth 26 francs 50 cents, find the number of francs that can be obtained for 1000 guineas.

9. A clock gains $7\frac{1}{2}$ min. per day, find the gain per minute.

10. How many days would it take to count 800 million coins, at the rate of 125 per minute?

11. Find the L. C. M. of 6, 8, 12, 18, 24, 27, explaining the process that you use.

12. Express in the smallest integers the ratio of $\frac{3}{1\frac{1}{3}}$ to $\frac{6\frac{1}{2}}{5}$.

13. After using $\frac{2}{5}$ of a cheese, $\frac{2}{3}$ of the remainder sold for £1 2s. $5\frac{1}{2}$d.; what was the whole cheese worth?

14. From the sum of $\frac{2}{5}$ of 7s., and $\frac{3}{4}$ of half a guinea, take $\frac{2}{5}$ of a guinea, expressing the result as the fraction of a moidore.

15. What number added to $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{2}{7}$ of $\frac{5}{8}$, will make a sum total of 5?

D.

1. Find the value of $3645 \times 700705 \div 37$.

2. How many acres in a field 202 yds. long, and 167 yds. broad?

3. A bar of gold, valued at £3 17s. $10\frac{1}{2}$d. per oz., is sold for £1429 0s. $1\frac{1}{2}$d.; what was its weight?

4. A sheet of letter-press contains 24 pages, of which $\frac{2}{3}$ are large type, having 32 lines to the page, and 54 letters in a line; and the remainder is small type, with 45 lines to the page, and 64 letters in a line, how many letters are there in the sheet?

5. Out of a yearly income of £1000, £737 17s. 6d. is spent for 35 years, how much will have been saved in that time?

6. In a foot race, where 50 yards start are given to A, the hindermost B gains 5 feet in every 50 yds.; where will the competitors be at the end of a mile?

7. The duty on tea is 2s. $2\frac{1}{4}$d. per lb., how much must pay the duty, to make 5 millions sterling?

8. How many plots of land, each 50 square perches, can be made out of a square mile?

9. The pendulum of a church clock vibrates 15 times in 4 minutes, how many vibrations in 24 hours?

10. Define proper, improper, and compound fractions, giving two examples of each.

11. Compare $\frac{11}{15}$, $\frac{2\frac{1}{3}}{7\frac{1}{3}}$, and $\frac{2}{5}$ of $\frac{7}{9}$.

12. A person receives £750 for $\frac{3}{16}$ of his share of a mine worth £10000; what fractional part of the whole was his share?

13. What is the ratio, expressed in integral numbers, between the sum and difference of $17\frac{2}{9}$ and $27\frac{1}{11}$?

14. If I gain $18\frac{3}{8}$s. in 15 guineas, how much is that in the pound?

15. Three men, A, B, C, can do a piece of work in 2, $2\frac{1}{2}$, and $3\frac{1}{3}$ hours respectively, how much of the work could be done in 20 minutes by them all working together?

DECIMAL FRACTIONS,

OR

DECIMALS.

91. It is known to all who understand *Numeration*, that where several figures are placed in a horizontal row, so as to form one number, the value of any figure depends upon its distance from the figure nearest to the right, or, as it is commonly called, from the units' place. Thus, if we take the number 6666, we know that the four figures, counting from left to right, have these values respectively—6000, 600, 60, 6; where we observe that the first 6 to the left has 10 times the value of the second—the second, 10 times the value of the third, and so on: or, going from right to left, each figure has one-tenth of the value of the one preceding it; in other words, any figure, when moved from right to left is multiplied by 10 every step, and when moved from left to right is divided by 10 every step. For example, in the number

$$\overset{\text{D}}{6}\ \overset{\text{C}}{6}\ \overset{\text{B}}{6}\ \overset{\text{A}}{6},$$

if I move 6 from A to D, or three places to the left, I in reality multiply it by $10 \times 10 \times 10$, or 1000, *i. e.* the 6 becomes 6000. Again, to change a 6 from C to A, or two places to the right, I divide it by 10×10, or 100; *i. e.* the figure which before represented 600 now represents 6.

92. Since, then, it has been shown that successive figures to the right are found by dividing by 10, let this

division be continued beyond the units' place; we ought, therefore, to have, as the value of the first figure to the right of the units' place, $\frac{1}{10}$th of 6, or $\frac{6}{10}$; of the next, $\frac{1}{10}$th of $\frac{6}{10}$, or $\frac{6}{100}$; so, of the next, $\frac{6}{1000}$; of the next, $\frac{6}{10000}$, &c. And when these four additional figures are placed to the right of the units' place, the entire number will be $6666\cdot\overset{EFGH}{6666}$, where a point has been set after the units' place, to show where the new figures commence. The entire number now consists of 6 thousands, 6 hundreds, 6 tens, 6 units (or 6), 6 tenths, 6 hundredths, 6 thousandths, 6 tenths of thousandths.

Also, observing $\cdot\overset{EFGH}{6666}$, it may be seen that the same rule holds that was true of the whole numbers, viz. that to move any figure from right to left is to multiply by 10 every step, and from left to right, is to divide by 10. For example: if I change the second 6 from F to H, or move it *two* places to the right, I change it from $\frac{6}{100}$ to $\frac{6}{10000}$, which $= \frac{6}{100} \div 100$; *i. e.* I have divided it by 10×10, or by ten *twice*.

Again, if I change the fourth 6 from H to E, or move it *three* places to the left, I change it from $\frac{6}{10000}$ to $\frac{6}{10}$, which $= \frac{6}{10000} \times 1000$; *i. e.* I have multiplied $\overset{H}{6}$ by $10 \times 10 \times 10$, or by 10 three times; hence the same law holds both on the right and left of the point.

DEF. If a number be multiplied by itself any number of times, it is said to be raised to a power. Thus, the multiplication of $2 \times 2 \times 2$ is otherwise expressed by saying that 2 is raised to the power of 3; and a small figure 3 placed to the right of the 2 and above the line (thus, 2^3) indicates or explains how many factors, 2, have been multiplied together. In like manner, 10^4 expresses the

multiplication of four factors, each 10, and is called the fourth power of 10.

This small figure is called the *Index*, or *Exponent*.

93. It appears that the figures on the right of the point in reality represent fractions, as $\frac{6}{10}$, $\frac{6}{100}$, $\frac{6}{1000}$, &c., all of which have as denominators either 10, or powers of 10; hence they are called *Decimal* Fractions, or Decimals. And the point which is placed to separate the whole numbers from the decimals is called the *decimal point*.

Obs. For the future, in speaking of Decimal Fractions, I shall use the single word *Decimals;* and for Vulgar Fractions, the word *Fractions*.

94. We have just seen (91) that in writing down the value which any single figure represents in the whole number 6666, viz. 6000, 600, 60, 6, we place as many ciphers at the right-hand of each 6, as will keep it in the place which it had in the number 6666: so also, in the number ·6666, if we wish to write down the value of each one of these four figures separately, we shall have to place as many ciphers to the left of each 6 as will keep it in the place which it had in the number ·6666, or at its proper distance from the *point*. Hence these four figures, when placed singly, would be

$$\cdot 6, \quad \cdot 06, \quad \cdot 006, \quad \cdot 0006; \qquad (X)$$

and, as in whole numbers we might go farther to the *left*, and have as the next figure 60000; so, by going farther to the *right* in the decimals, I should have as the next figure ·00006.

Comparing the four quantities in line (X) with the value which we have shown to be due to them, viz. $\frac{6}{10}$, $\frac{6}{100}$, $\frac{6}{1000}$, &c., we have this connection,

$$\frac{6}{10} = \cdot 6 \qquad \frac{6}{100} = \cdot 06 \qquad \frac{6}{1000} = \cdot 006 \qquad \frac{6}{10000} = \cdot 0006$$

where it will be observed, that the number of *ciphers* in the den^r of the left-hand side equals the entire number of figures after the decimal point, whether ciphers or not, on the right-hand side.

95. Taking this fact as proved when there is but one figure in the num^r of the decimal frac^n, I shall now show that this is true, whatever be the number of figures in the num^r of the frac^n; for example, that $\frac{675}{10000} = \cdot 0675$, where there are four *ciphers* in the left-hand den^r, and four *figures* after the decimal point.

$$\text{For } \frac{675}{10000} = \frac{600}{10000} + \frac{70}{10000} + \frac{5}{10000}$$
$$= \frac{6}{100} + \frac{7}{1000} + \frac{5}{10000};$$

and these, by what has just been shown, in last Article,

$= \cdot 06 + \cdot 007 + \cdot 0005$

$= 6\text{-hundredths} + 7\text{-thousandths} + 5\text{-tenths of thousandths}$

$= \cdot 0675$;

i. e. $\frac{675}{10000} = \cdot 0675$, which was the required result.

Hence any frac^n, having as a den^r any power of 10, can be immediately written as a decimal, by writing down the num^r only, and so placing the decimal point that it shall have as many figures on its right-hand, as there are ciphers in the den^r of the given frac^n. And if the num^r does not contain as many figures as there are ciphers in the den^r,—that is, as many as it is necessary to have after the point, the amount must be made up by placing as many ciphers between the point and the figures taken from the num^r as shall complete the desired number.

$$\text{Thus, } \frac{3275}{1000} = 3\cdot 275 \qquad \frac{743}{100000} = \cdot 00743.$$

In the latter Ex. I find it necessary to place two ciphers between the point and the 743, to make the number of

figures to the right of the point equal to the number of ciphers in the left-hand den^r.

The pupil should now be able to read any decimal either in one sum, or express it in terms of the several parts of which it is composed. Thus, 3·275 may be read in terms of its several den^{ns}, viz. 3, and 2 tenths, 7 hundredths, and 5 thousandths: or 3, and 275 thousandths; and here it will be observed that when a decimal is expressed in one den^n, that den^n will be the lowest contained. Thus in reading ·00743, since the 3 stands for hundredths of thousandths, therefore ·00743 stands for 743 hundredths of thousandths.

96. The position of the decimal point determines the value of every figure both on the right and left of it, that is, both of the whole numbers and the decimals. Therefore, to move the point to the right has the same effect as moving all the figures to the left; and to move the point to the left, is equal to moving the figures to the right.

Now, it has been shown (92) that to move a figure one place to the left is to multiply it by 10; therefore, if in any number containing a decimal point, I move the point one place to the right, I in reality multiply every figure in the number, and therefore the entire number, by 10: similarly, if I move the point to the left one place, I divide it by 10. Hence, if I wish to multiply a number containing a decimal point by 10, 10^2, 10^3, &c., I move the point 1, 2, 3, &c. places to the right; and if I wish to divide the number by 10, 10^2, 10^3, &c., I move the point 1, 2, 3, &c. places to the left.

Ex. 3275·468. (Y)

Moving the point two places to the right, the number becomes 327546·8; and it will be found that any figure has

now 100 times the value that it had before. The 5 formerly stood for 5 *ones*, or 5, but now for 5 ***hundreds***, or 500; the 6 for 6 hundredths, or $\frac{6}{100}$, but now for 6, or $\frac{6}{100}, \times 100$; that is, each figure has been multiplied by 100, merely by moving the point two places to the right.

Next, let the point be moved three places to the left, and the number becomes $3{\cdot}27546\dot{8}$; and we then see that the 7, which in (Y) was 70, is now only $\frac{7}{100}$, or $\frac{70}{1000}$; that is, has one-thousandth of its former value: so, also, the 3, which before represented 3000, now represents only 3; hence it appears that the entire number has been divided by 1000, merely by moving the point three places to the left.

Exs. 20.

I. Multiply 8·034 by 10, 10,000 100,000, and 1,000,000 successively.
II. Divide 175·04 by 100, 100,000, 1,000,000 and 10, successively.
III. Multiply ·005 by 10^3, 10^5; and divide it by 10, 10^2, 10^4.

CONVERSION OF VULGAR FRACTIONS INTO DECIMALS.

(Thrower, Cases I. and III.)

97. In the use of Decimals we shall find it necessary to know how to convert decimals into fractions, and fractions into decimals. And it will be seen that there are two classes of fractions, one containing those which can be exactly expressed as decimals, and the other, such as cannot. We shall presently show the ground upon which this variety rests; and shall first treat of the former class, which includes all fractions which, *when in their lowest terms*, have, as den^{rs}, numbers which contain the figures 2 and 5 as their only factors. These may be called *convertible* fractions.

98. We have seen that a frac^{n} whose den^{r} is a power of

10, can immediately, by inspection, be converted into a decimal. Hence, to bring a fraction into a decimal, I must first, if possible, reduce it to a fraction with such a denominator.

And since fractions can be changed into equivalent fr^{ns} of different den^{rs}, only by multiplying or dividing both num^{r} and den^{r}, my object will be to find some multiplier or divisor which shall transform the den^{r} of any proposed fr^{n} into the required form, 10, 100, 1000, &c.

Any den^{r} which contains an equal number of both 2's and 5's, will evidently contain as many factors 10, as 2 and 5, and therefore be of the form 10, 100, 1000, &c. Take for example as a den^{r} $2 \times 2 \times 5 \times 5 = 10 \times 10 = 100$: and when a fr^{n} with such den^{r} is converted into a decimal, there will plainly be as many decimal places as there are 2's or 5's.

Next, let the den^{r} contain 2's only, or be a power of 2, as $\frac{3}{8}$ or $\frac{3}{2^3}$. Now, if I multiply both num^{r} and den^{r} by some power of 10, and then divide both num^{r} and den^{r} by the original den^{r} 8, the resulting fr^{n} will have a den^{r} of the required form: but if the new num^{r} can be divided exactly by 8, it follows that the multiplier must contain all the factors of 8, *i. e.* $2 \times 2 \times 2$; and since this multiplier contains only 10's, and each 10 contains one 2, therefore I must have three factors 10 to contain the above three 2's; *i. e.* my multiplier must be $10 \times 10 \times 10 = 1000$. Using, then, this multiplier, and dividing by the 8, I have the whole process as follows:

$$\frac{3}{8} = \frac{3000}{8000}\left(= \frac{\frac{3000}{8}}{1000}\right) = \frac{375}{1000} = \cdot 375 \qquad \text{(A)}$$

and in dividing 3000 by the 8, it is found that the division will not terminate until I have used the 3 ciphers, as was foreseen.

We have just now multiplied numr and denr of $\frac{3}{8}$ by 1000 and divided by 8; that is, we have multiplied by $\frac{1000}{8}$, or 125: if, therefore, we at once multiply by 125, we ought to obtain the same result. The work would then be

$$\frac{3}{8}=\frac{3\times 125}{8\times 125}=\frac{375}{1000}=\cdot 375.$$

99. In working Exs. of this class, a pupil should at first write out the operation as in line (A), omitting the frn in brackets; but in practice the work may be performed mentally, by the method of Short Division.

Since 8 is a divisor, I commence as follows:—8 in 3 units gives no units; I must therefore bring these units into tenths, and say, 8 in 30 tenths, which gives 3 *tenths* as quotient, and 6 tenths over; and since 6 tenths = 60 hundredths, 8 in 60 hundredths gives 7 *hundredths*, and 4 hundredths over, or 40 thousandths; 8 in 40 thousandths gives 5 *thousandths*; and there being no remainder, the division terminates: and collecting the three quotients, viz. 3 tenths, 7 hundredths, and 5 thousandths, we plainly have $\frac{3}{8}=\cdot 375$.

In Simple Short Division, as soon as I know where to put the first quotient, I can write down the others in order; so here, as soon as I find that the first quotient consists of tenths, I can place the other quotients without further consideration. But in determining the first quotient there is a liability to error; for if a pupil takes the frn $\frac{3}{80}$, and begins to divide by 80, he may incautiously cut off the ciphers from divisor and dividend, as in Simple Division: but, by placing the frn thus

$$\begin{array}{r} 80)\ \underline{3\cdot 0000} \\ \cdot 0375 \end{array}$$

it will be found that if I cut off a cipher at the end of the

dividend, I remove a figure which has no value; whereas, to take 0 from the 80 diminishes it tenfold: I must therefore say, 80 in 3 units gives as quotient 0 units; so also, since 3 units = 30 tenths, 80 in 30 tenths gives 0 tenths. Again, 30 tenths = 300 hundredths, and 80 in 300 hundredths gives 3 hundredths; hence 0 must be put as quotient under the tenths, and 3 under the hundredths; the remaining quotients will then fall into their proper places by mere Simple Division.

But in examples like this, where the den^r contains a power of 10, it is best to divide by the other factors first, and then divide by that power of 10. In the Ex. $\frac{3}{80}$, we should divide both num^r and den^r by 8, and then divide by 10, by inspection. Thus

$$\frac{3}{80}=\frac{\cdot 375}{10}=\cdot 0375$$

$$\text{So, also, } \frac{3}{8000}=\frac{\cdot 375}{1000}=\cdot 000375 \qquad \text{by (96)}$$

100. If the den^r of the fr^n be a composite number larger than 12, we may perform the division by breaking the den^r into factors, and dividing by them successively. For example

$$\frac{5}{32}=\frac{1\cdot 25}{8} \qquad (B)$$
$$=\cdot 15625.$$

where the latter fraction in (B) was obtained by dividing num^r and den^r by 4.

A fraction whose den^r has 5's only as its factors, must be reduced to a decimal by a process precisely similar to the one exhibited in (A); and the power of 10, which is to be used as a multiplier of num^r and den^r, must, by the same reasoning as in (98), have as many factors 10 as there are factors 5 in the den^r.

Ex. $\frac{3}{25}$. Here there are two factors 5 in the den^r; therefore I shall have to multiply by 100, and divide by 25, *i. e.* the div^n will terminate when I have used 2 ciphers in the num^r.

$$\text{Thus, } \frac{3}{25} = \frac{\cdot 6}{5} = \cdot 12$$

$$\text{So also } \frac{3}{12500} = \frac{\cdot 6}{2500} = \frac{\cdot 12}{500} = \frac{.024}{100}$$

$$= \cdot 00024 \qquad \text{by (96)}$$

Exs. 21. Express as decimal fractions

I. $\frac{3}{10}$, $\frac{11}{1000}$, $\frac{19}{100}$, $\frac{15}{100{,}000}$, $\frac{1001}{10}$

II. $\frac{3}{5}$, $\frac{7}{5}$, $\frac{1}{2}$, $\frac{9}{4}$, $\frac{11}{8}$, $\frac{17}{16}$, $\frac{18}{25}$, $\frac{3}{500}$

III. $\frac{7}{128}$, $5\frac{3}{64}$, $11\frac{18}{625}$, $8\frac{1}{8}$ of $11\frac{3}{5}$, $7\frac{1}{80}$ of $5\frac{1}{16}$

101. Coming now to what we may call the *inconvertible* fractions, *i. e.* those which, when reduced to lowest terms, contain other factors than 2 and 5 in the den^r, we see at once, that since a power of 10 is the only multiplier which can render the den^r of the required form; therefore any factors in the den^r, as 3, 7, 9, &c., which are not contained in 10 cannot be cancelled out, as the 2's and 5's were in (98): hence the division, such as in (A), will not terminate. But since the successive quotients diminish tenfold in value every step, so that the eighth to the right of the point represents hundredths of millionths, we generally pursue the division till we obtain 7 figures in the decimal, and then consider the remaining quotients as too small in value to be much appreciated.

Moreover, whatever be the factor which cannot be cancelled out of the den^r, we know that in dividing by any divisor, the remainder must always be at least one less than the divisor: therefore if any divisor, as 7, be the factor which causes the division not to terminate, there can be

x remainders; and hence, when 7 divisions have been rmed, one of these six remainders must come over , or recur: and if the remainder, which, of course, nces the dividend, recurs, the quotients will also , and we shall then have the same set of figures ring, as it is termed.

. To convert $\frac{6}{7}$ into a decimal.

orking mentally, as in (99), I have

$$\frac{6}{7} = \cdot 857142857142, \&c. \qquad (C)$$

it appears that the figures 857142 will recur, to ver length the division may be carried: hence such als are called *Recurring Decimals*, and the set of ents which recurs is called a *Period.** If the period st of but one figure, that single figure is written with (·) over it in place of the whole decimal; but if be more figures than one, the period is written with over the first and last of its figures.

$$\text{Thus, } \frac{6}{7} = \cdot\dot{8}5714\dot{2}$$

$$\frac{1}{3} = \cdot 333 \ldots\ldots \text{ or } = \cdot\dot{3}$$

$$\frac{1}{9} = \cdot 1111 \ldots\ldots \text{ or } \cdot\dot{1}$$

$$\frac{3}{22} = \cdot 13636 \ldots\ldots$$

$$\text{or } \cdot 1\dot{3}\dot{6}.$$

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rving the line (C) it may be noticed that whenever a fr^n has 7 for its den^r, the decimal obtained will consist of the figures 857142; but this period will not egin with the figure 8, but with some other of the above 6 figures; thus 571 &c., where it will be observed that the period is the same as in (C), except egin with the 4; and if these figures are retained in the memory, the whole written down at once, as soon as the first figure has been obtained by division. ever, when the den^r is 7, the num^r be already a recurring decimal, the above on will not hold, because the additional figures at the end of the num^r are not s in the above fr^ns $\frac{6}{7}$ and $\frac{3}{7}$.

Obs. The places which are occupied by the figures to the right of the decimal point are often called *decimal places.* Thus, in the Ex. $\frac{6}{7}$, we say that the division was carried to six places before the figures began to recur.—(See Appendix, Art. Circulating Decimals).

The following equalities are worth remembering.

$$\frac{1}{8} = \cdot125 \qquad \frac{1}{4} = \cdot25$$

$$\frac{3}{8} = \cdot375$$

$$\frac{1}{2} = \cdot5$$

$$\frac{5}{8} = \cdot625$$

$$\frac{7}{8} = \cdot875 \qquad \frac{3}{4} = \cdot75$$

Their utility appears as follows:—if in converting a fraction to a decimal I have a divisor 8, and I come to the last figure in the dividend, one of the above equalities will enable me to write down the remaining quotients immediately.

Thus, if I have to reduce $\frac{75}{8}$ to a decimal, I find that after one quotient 9 has been obtained, 3 is my last dividend; hence, since $3 \div 8 = \cdot375$, I can write $\frac{75}{8} = 9\cdot375$, without *performing* the division for the last three quotients. Similarly, $\frac{11\cdot7}{8} = 1\cdot4625$, where the last three figures were obtained, as before, by remembering the value of $5 \div 8$. So also, $\frac{75}{4} = 18\cdot75$,—the last two quotients being written without dividing.

Exs. 22. Express as decimal fractions

I. $\frac{1}{6}$, $\frac{1}{7}$, $\frac{3}{7}$, $\frac{2}{11}$, $\frac{8}{13}$, $\frac{7}{9}$, $\frac{1}{3}$ of $1\frac{3}{8}$

II. $\frac{1}{3}$, $\frac{5}{9}$, $\frac{11}{7}$, $\frac{13}{28}$, $\frac{17}{65}$, $\frac{4}{27}$, $\frac{3}{56}$, $\frac{2}{3}$ of $1\frac{1}{18}$

III. $8\frac{5}{12}$, $17\frac{4}{33}$, $375\frac{7}{55}$, $\frac{1915}{390}$

TO CONVERT TERMINATING DECIMALS INTO VULGAR FRACTIONS.

(Thrower, Case III.)

102. It will be easily seen that a number expressed as a decimal can be immediately exhibited as a vulgar fraction.

For, since it has been shown (95) that $\frac{675}{10000} = \cdot0675$, therefore we can reverse the process, and say that $\cdot0675$, when converted into a frⁿ, becomes $\frac{675}{10000}$; and the correctness of this conversion may be proved thus:

$$\cdot0675 = \frac{6}{100} + \frac{7}{1000} + \frac{5}{10000}$$

$$= (\text{when reduced to L. C. D.})\ \frac{600 + 70 + 5}{10000}$$

$$= \frac{675}{10000}. \qquad \text{(D)}$$

$$\text{So also, } 3\cdot0275 = 3 + \frac{0}{10} + \frac{2}{100} + \frac{7}{1000} + \frac{5}{10000}$$

$$= \frac{30000 + 200 + 70 + 5}{10000} = \frac{30275}{10000}, \qquad \text{(E)}$$

a fraction greater than 1, as might have been foreseen, because 3·0275 is partly a whole number and partly a decimal.

Def. In a whole or mixed number, the part which is not fractional is called *Integral*, or an *Integer*.

103. In (D) we observe that the num^r of the vulgar fr^n into which we have changed the decimal, before it is reduced to lowest terms, consists of the figures in the given decimal, excluding ciphers; and the den^r is 1, followed by as many ciphers as there are decimal places.

But in (E), where a cipher occurs in the given decimal 3·0275, it remains in the num^r of the equivalent improper fr^n; and the den^r is formed as before.

The above fractions may, of course, be reduced to lowest terms; but I have left them in their present shape, in order to show the connection between the given decimal and the fractional form into which it could be converted by inspection. Reducing them, and writing the Exs. as they should be exhibited, we have

$$\cdot0675 = \frac{675}{10000} = \frac{27}{400}; \quad \text{and } 3\cdot0275 = \frac{30275}{10000} = \frac{1211}{400} = 3\tfrac{11}{400}.$$

If the given number be partly a whole number and partly a decimal, we may leave the integral part unaltered, and then the resulting frⁿ will appear at once as a mixed number.

$$\text{Thus, } 76{\cdot}0725 = 76\tfrac{725}{10000} = 76\tfrac{145}{2000} = 76\tfrac{29}{400}$$

Exs. 23. Convert into vulgar fractions, in their lowest terms,

I. ·05, 10·73, 115·008, ·0001, 1·005, ·343.

II. 12·01, 10·008, ·00725, 135·55, ·10505, 9·99.

TO CONVERT RECURRING DECIMALS INTO VULGAR FRACTIONS.

(Thrower, Case IV.)

104. Though at first it may be thought that a ***non-terminating*** decimal cannot be accurately represented by a frⁿ; yet, since every recurring decimal is formed from one of that class of fractions which was discussed in (101), therefore every such decimal can of course be made to resume the shape from which it was derived: and the accuracy which was lost in the change from a fraction to a decimal is restored by this reconversion to the original fraction.

105. Circulating Decimals are of two kinds; one in which the whole of the figures repeat, as

$$\cdot 363636\ldots\ldots\ldots \text{or } \cdot\dot{3}\dot{6};$$

and the other, in which some of the figures to the right of the point are not repeated. These figures, of course, always stand to the left of the circulating part, because when a decimal once begins to circulate, it continues to do so. An Ex. of this kind is

$$\cdot 754365365\ldots\ldots\ldots \text{or } \cdot 754\dot{3}6\dot{5}.$$

The former kind is called a *Pure*, and the latter a *Mixed* Circulating Decimal.

106. The best method of working Exs. under this case will not be intelligible without an acquaintance with one or two facts in algebra. A letter of the alphabet may be used to represent a quantity, the value of which we have to find, as in (73). Thus, I may let F stand for or represent the fraction which is equivalent to any circulating decimal; and it is for me to ascertain the value of this F in any particular example.

Also, a number placed just before this F, and with no sign connecting F and the number, is a multiplier of the F: thus, 10F, $\frac{2}{3}$F, mean ten times F, $\frac{2}{3}$ times F, or $\frac{2}{3}$ of F; and they have the same value as though they were written $10 \times F$, $\frac{2}{3} \times F$. F repeated once is not written 1F, as we should write 1s., but only F. These multipliers 10, $\frac{2}{3}$, and 1, are called *coefficients*. Moreover, since in the quantity 10F, F is the den^n, and the 10 tells us how many times F is taken, just as the 7 in 7s. tells us how many shillings are taken, therefore I can add to this 10F, or subtract from it, any number of quantities of the same den^n, *i. e.* any number of times F.

$$\text{Thus, } 10F + 3F = 13F \qquad 10F - 3F = 7F$$
$$\text{and } 100F + F = 101F \qquad 100F - F = 99F.$$

107. I now proceed to find the fr^n which is equivalent to any circulating decimal, as

Ex. I. $\cdot\dot{3}\dot{6}$ or $\cdot363636\ldots\ldots$

Let $F = \cdot363636\ldots\ldots$ (G)

Now if I multiply both sides of this equation by the same number, the equality will not be disturbed. My object is to move the point two places to the right in the decimal ·363636......, so that it shall become 36·3636......This is done by multiplying both sides by 100; the left-hand side

therefore becomes $100 \times F$, or $100F$; hence we have

$$100F = 36{\cdot}3636\ldots\ldots\ldots \quad (H)$$
$$\text{also } F = \cdot 363636\ldots\ldots\ldots \quad (G)$$

Subtracting (G) from (H), we observe that on the left-hand side, F or 1F taken from 100F leaves 99F as in (106); and on the right-hand side, since both decimal parts commence alike and go on for ever, their difference is 0; and after sub^n there remains only the integral number 36; hence we have

$$99F = 36$$

and dividing both sides by 99, that is, transferring 99 into the right-hand den^r as in (71) and (72), we find

$$F = \frac{36}{99} \qquad (I)$$
$$= \frac{4}{11}$$

By observing (I) we learn that the fraction which represents the value of the circulating decimal $\cdot\dot{3}\dot{6}$ has for its num^r the figures in the period, viz. 36, and for the den^r, as many 9's as there are figures in that period.

In accordance with this rule we may therefore, by inspection, find the value of any circulating decimal which contains only such figures as do repeat.

$$\text{Thus, } \cdot\dot{6} = \frac{6}{9} = \frac{2}{3}$$

$$\text{and } \cdot\dot{1}4\dot{7} = \frac{147}{999} = \frac{49}{333}.$$

but every pupil ought to work some Exs. out fully as in the beginning of this article.

108. Again, noticing (H) and (G) we see that the figures to the right of the decimal point are the same in both lines; and this result was produced by so multiplying

both sides of the equation (G), that the point might be made to pass over one period; in this case it had to be moved two places to the right; that is, I had to multiply by 10×10, or 100. If the period had contained four figures, I should have multiplied by 10^4, or 10000. I will take as an Example

Ex. II. To express $\cdot\dot{2}21\dot{6}$ as a fraction.

$$\text{Let } F = \cdot 22162216 \ldots\ldots\ldots \quad (K)$$

$$\text{therefore, } 10000F = 2216 \cdot 2216 \ldots\ldots\ldots \quad (L)$$

Subtracting (K) from (L), we have

$$9999F = 2216;$$

$$\text{therefore, } F = \frac{2216}{9999}:$$

and it is plain that this result might have been written down at once by taking as num^r of the required fr^n the period 2216, and as den^r a number consisting of as many figures 9 as there are figures in this period.

109. We now proceed to convert a mixed circulating decimal into a fraction. The nature of the process is the same as in the last article; but one step more is required, the reason of which will appear in the operation.

Ex. III. To convert $\cdot 32\dot{7}1\dot{5}$ into a fraction.

$$\text{Let } F = \cdot 32715715 \ldots\ldots\ldots \quad (M)$$

Then, as before, (107), carrying the decimal point over the period, or five places to the right—*i. e.* multiplying both sides by 100000, I have

$$100000F = 32715 \cdot 715 \ldots\ldots\ldots (N)$$

It now appears that I cannot subtract (M) from (N), so as to get rid of the decimal parts, because the quantities to the right of the point are not the same: but I see that I can very readily obtain another equation which shall have the decimal part the same as in (N); for if in (M) I

multiply both sides by 100, *i. e.* move the point two places to the right, or over the non-recurring part alone, I shall have

$$100F = 32{\cdot}715715\ldots\ldots\ldots \quad (O)$$

therefore, subtracting (O) from (N) and recollecting (108) I have

$$[100000F - 100F = 32715{\cdot}715\ldots\ldots - 32{\cdot}715715\ldots\ldots] \quad (P)$$

$$\text{or } 99900F = 32715 - 32;$$

$$\text{therefore, } F = \frac{32715 - 32}{99900} \quad (Q)$$

$$= \frac{32683}{99900}.$$

110. In the line (P) I learn that the larger coefficient of F has as many ciphers as there are figures from the point to the end of the first period; and the smaller coefficient has 2 ciphers, *i. e.* as many as there are figures in the non-recurring part: also, when the subtraction is *performed* in the next line, the coefficient of F has as many ciphers as there are figures in the non-recurring part: and the remaining figures to the left are all nines, and as many in number as there are figures in the period. Hence, observing the right-hand side of (Q), I learn that in converting a mixed circulating decimal into a fraction I obtain as num^r, "the figures of the given decimal to the end of the first period—the figures in the non-recurring part;" and as den^r, as many figures 9 as there are figures in the period, followed by as many ciphers as there are figures in the non-recurring part. If there be an integer in the given recurring decimal, I may omit it while finding the value of the recurring part; and afterwards by inserting it I shall have the resulting fraction a mixed number; or I may retain it throughout, as in Ex. V. below, and the result will be an improper fraction.

I will write down one more Ex. of each kind of circulating decimals, in the form in which they ought to be worked.

Ex. IV. Find the fraction equivalent to $\dot{6}3\dot{9}$.

$$\text{Let } F = \cdot 639639639 \ldots\ldots\ldots \quad (\text{I.})$$

multiplying by 1000,

$$1000F = 639\cdot 639639 \ldots\ldots\ldots \quad (\text{II.})$$

Subtracting (I.) from (II.),

$$999F = 639,$$

$$\text{or } F = \frac{639}{999} = \frac{213}{333} = \frac{71}{111}.$$

Ex. V. Find the fraction equivalent to $2\cdot 0\dot{3}4\dot{5}$.

$$\text{Let } F = 2\cdot 0345345 \ldots\ldots\ldots \quad (\text{III.})$$

multiplying by 10000,

$$10000F = 20345\cdot 345 \ldots\ldots\ldots \quad (\text{IV.})$$

Again, multiplying (III.) by 10, $10F = 20\cdot 345345 \ldots\ldots\ldots$ (V.)

and subtracting (V.) from (IV.),

$$9990F = 20345 - 20$$

$$\text{therefore, } F = \frac{20325}{9990} = \frac{6775}{3330} = \frac{1355}{666}$$

$$= 2\tfrac{23}{666};$$

or omitting the integral part, as described above, I have

$$\text{the fractional part} = \frac{345}{9990} = \frac{69}{1998} = \frac{23}{666},$$

therefore, the entire quantity $= 2\frac{23}{666}$, as before; and this is the better method.

Exs. 24. Convert into vulgar fractions, in their lowest terms,

I. $\cdot\dot{1}$, $\cdot\dot{2}7\dot{3}$, $\cdot\dot{0}\dot{9}$, $\cdot\dot{0}10\dot{1}$, $15\cdot\dot{0}7\dot{5}$, $\cdot\dot{1}4\dot{2}85\dot{7}$.

II. $\cdot 1\dot{6}$, $\cdot 0\dot{2}\dot{7}$, $130\cdot\dot{2}8571\dot{4}$, $\cdot 000\dot{1}\dot{9}$, $\cdot 14285\dot{7}$, $35\cdot\dot{0}0\dot{9}$.

III. $\cdot 08451\dot{6}$, $60\cdot 010\dot{1}\dot{4}$, $100\cdot 0\dot{3}\dot{6}$, $\cdot 16\dot{7}843\dot{2}$. $35\cdot 00\dot{9}$, $3500\cdot\dot{9}$.

ADDITION.

(Thrower, Case V.)

111. Since we already know that quantities cannot be added together unless they be of the same den^n, so in the addition of decimals, we must take care to add tenths to

tenths, hundredths to hundredths, and so on. And this will be readily done, if we so place all the numbers under one another, that the decimal points may be in a vertical row. By observing the Ex. worked below, we notice that the tenths are all placed under one another, as are also the hundredths, thousandths, &c.

Ex. I.

$$\begin{array}{r} 732\cdot416 \\ \cdot084 \\ \cdot000007 \\ 93\cdot268 \\ 2708\cdot4153 \\ \hline 3534\cdot183307 \\ \hline \end{array}$$

Adding up the third row to the right of the point, which consists of thousandths, I find that it amounts to 23 thousandths, or $\frac{23}{1000}$, which $= \frac{20}{1000} + \frac{3}{1000} = \frac{2}{100} + \frac{3}{1000}$; I therefore put down the 3 thousandths, and carry the 2 hundredths to the next column, which consists of hundredths; and since this step is just such as would be performed in Simple Addition, it is plain that all the rest of the work may be performed by that Rule.

112. But if in the quantities to be added there are circulating decimals, we may either, 1st, convert into a frⁿ each circulating decimal, and having found the sum of all these fractions, reduce the result to a circulating decimal: or, 2ndly, (and this method is the better,) write down the recurring decimals at length, to as many places as will include twice the longest period; observe where there are two vertical columns alike, though not necessarily close to each other; commence the addition three or four places to the right of the second of these two similar columns, and complete the addition, as before: the sum found will be seen to be carried far enough to enable a pupil to detect the period in the answer.

Ex. II. Find the sum of $\cdot\dot{7}1428\dot{5} + \cdot9\dot{2}8571\dot{4} + 20\cdot0\dot{9}2\dot{5} + 5\cdot4\dot{0}4761\dot{9}$.

Writing these at length, and working according to the above directions, I have

$$\begin{array}{r} \;\;\text{A}\text{B} \\ \cdot714285714285.... \\ \cdot9285714285714.... \\ 20\cdot0925925925925.... \\ 5\cdot4047619047619.... \\ \hline 27\cdot14021164020 \\ \hline \end{array}$$

Here I observe that the first two similar vertical columns are the second and the eighth, marked (A) and (B). Commencing the addition at the third column beyond (B), I find that I have figures enough in the result to show that the sum of the given circulating decimals is $27\cdot1\dot{4}0211\dot{6}$.

Exs. 25. Find the value of

1. 18·325 + ·0007 + 70·1 + 358 + 3·04705 + 1000·06.
2. 347·859 + ·010401 + 639 + 2·573 + 11·01115 + ·32784.
3. 1·000009 + 45 + 3845·1 + 75·6832 + 10·01 + ·04311.
4. 7600 + 3·1009 + 473·842691 + ·07 + ·00001 + 1·1.
5. $3458 + 5\cdot143\dot{2}8\dot{5} + \cdot\dot{0}7\dot{5} + 145\cdot\dot{2}\dot{7} + \cdot\dot{8}7516\dot{9}$.
6. $\cdot\dot{3} + \cdot\dot{0}\dot{3} + 145\cdot27\dot{3}4\dot{5} + 3\cdot\dot{0}0\dot{9} + 6\cdot\dot{1}4285\dot{7}$.
7. $\cdot\dot{6}8\dot{4} + 1\cdot\dot{5}8764\dot{9} + 3\cdot\dot{8}41007\dot{7} + \cdot\dot{0}\dot{7}$.
8. $1\cdot\dot{6} + 19\cdot3485\dot{4} + \dot{0}\cdot2\dot{7} + 5\cdot34785\dot{6} + 111\cdot1 + \cdot0\dot{3}\dot{5}$.

SUBTRACTION.

(Thrower, Case VI.)

113. As Addition of Decimals was shown to be only an extension of Simple Addition, so Subtraction of Decimals is of the same nature with Simple Subtraction; but we must take care how we subtract, when the lower line contains more decimal places than the upper.

Ex. I. Find the value of 18·0426 − 2·005417.

Placing the two quantities one under the other, so that the points are in a vertical row, we shall, as in Addition, have tenths under tenths, &c.

As there are no figures in the 5th and 6th places in the upper line, I may place ciphers there; and since there are no figures to the right of these ciphers, the value of the decimal will plainly remain unaltered.

(A)	
18·042600	600
2·005417	417
16·037183	183

If we now take the last three figures in each row as whole numbers, so as to form a Simple Subtraction Sum, we find the difference to be 183: so also, since the 600 and 417 in (A) both represent millionths, their difference is 183 millionths: similarly the 42 and 5 are both thousandths, and their difference = 37 thousandths: hence we see that if the subtraction be performed in (A) as in a common Ex. in Subtraction, the result will be correct.

114. If an Ex. in Subtraction contain circulating decimals, we must, as in Addition, write down the periods sufficient to contain twice the longest period, and commence the subtraction about five places beyond the latter one of the two similar rows; the remainder which will recur will be found to contain figures enough to enable us to detect the period.

Ex. II. Find the value of $24\cdot\dot{4}0\dot{7} - 5\cdot98\dot{0}7697\dot{3}$.

Here, writing five of the periods in the upper line, and two in the lower line, we have

$$\begin{array}{r} \mathrm{A}\mathrm{B} \\ 24\cdot407407407407407\ \ldots \\ 5\cdot98076973076973\ \ldots \\ \hline 18\cdot4266376766377 \end{array}$$

The columns marked (A) and (B) are the first pair which are similar; I therefore commence the subtraction at the fifth row beyond (B): and the remainder, which contains the circulating decimal, is evidently $18\cdot42\dot{6}6376\dot{7}$.

By observing the Ex. in Addition of circulating decimals, I find that the figures which should form the second period are somewhat below the value of those in the first period; but by commencing with the addition a little more to the right, I should have figures to *carry*, as it is called, which would have made the second period correct as well as the first. So in the Ex. just worked in Subtraction, the figures which should form the second period are too large; but by commencing the Subtraction a few more places to the right, the process of *borrowing* would make the figures of the second period quite correct.

Exs. 26. Find the value of

I. $3\cdot5 - \cdot075$; $175\cdot8 - 1\cdot0024$; $\cdot715 - \cdot70451$.

II. $1\cdot325 - \cdot4736$; $10\cdot01 - 9\cdot875$; $1\cdot\dot{3}\dot{6} - 75$; $\cdot\dot{3} - \dot{1}$.

III. $\cdot\dot{3} - \cdot07$; $3\cdot0758 - 1\cdot\dot{2}7\dot{3}$; $13\cdot4\dot{0}8\dot{9} - 6\cdot85\dot{3}7492\dot{1}$.

MULTIPLICATION.

(**Thrower, Case VII.**)

115. The rule for the multiplication of decimals is to be found by multiplying the vulgar fractions equivalent to

any given decimals, and then observing the nature of the product so obtained.

Ex. I. To find the product of 3·275 and 18·03.

$$\text{We have } 3{\cdot}275 \times 18{\cdot}03 = \frac{3275}{1000} \times \frac{1803}{100}$$

$$= \frac{3275 \times 1803}{100000} = \frac{5904825}{100000} \qquad \text{(R)}$$

$$= 59{\cdot}04825.$$

From (R) we learn that the product of the given decimals is a decimal fraction which has for its numerator the product of the given numbers treated as integers, and for its denominator 1, followed by as many ciphers as there are decimal places in *both* the multiplier and multiplicand; and such a fraction is of course immediately (95) convertible into a decimal, which shall have as many decimal places as this fraction has ciphers; and this conversion is performed in the next line. Hence, as this Ex. differs not from any other which contains only terminating decimals, we conclude that the product of two such decimals is found by multiplying them together as in whole numbers, and pointing off in the product as many places as are found in both multiplier and multiplicand.

116. Also, if it be required to multiply together more than two numbers involving decimals, it will be found that the same rule must be observed. For suppose that four numbers were given, whose product was required: the product of the first pair might be found, as above; then this product and the third number might be multiplied: and, lastly, this second product and the fourth number: and so on for any number of quantities; hence we may find the product of any number of decimal quantities, as in the previous Example.

Ex. II. Find the value of ·0095 × 2·07 × 7·06 × ·0081.

$$\text{The product} = \frac{95}{10000} \times \frac{207}{100} \times \frac{706}{100} \times \frac{81}{10000}$$

$$= \frac{95 \times 207 \times 706 \times 81}{1000\,000\,000\,000} = \frac{1124562690}{1000\,000\,000\,000}$$

$$= {\cdot}001124562690$$

where the number of decimal places, viz. $12 =$ sum of the numbers of places in the four given quantities.

117. When a few Exs. have been worked illustrating the truth of the principles here stated, we may then work all similar Exs. as in Sim^e Mult^n of whole numbers, and point off as many decimal places in the product, as there are in all the numbers to be multiplied together.

Thus Ex. I. would have been commonly worked as follows;

```
   3·275
   18·03
   -----
    9825
 262000
 3275
--------
59·04825
========
```

Much instruction may also be derived from working an Ex. in the following manner:

Ex. III. To find the product of 271·405 and 93·6854.

```
  271·405
   93·6854
 ----------
  814·215
  162·8430
   21·71240
    1·357025
     ·1085620
 24426·45
--------------
 25426·6859870
==============
```

Placing the multiplier under the multiplicand in any position whatever, I commence multiplying by the figure in the units' place, viz. 3. Since then I am now merely repeating every figure in the multiplicand 3 times, therefore every figure when multiplied will give a product of the same *kind* as itself, and that product will occupy the same place with respect to the point as it did in the mult^d; hence the product will clearly be 814·215.

If, now, I multiply by the 6, which is in the *tenths'* place, the product will be ten times less than it would have been, had the 6 been in the *units'* place; hence I place every figure in this product one place farther to the *right* than in the previous line; so, also, the product by the 8 will be *two* places to the right, by the 5, will be *three* places, &c.

Again, since the 9 in the multiplier represents 90, I place the product obtained by multiplying by the 9 one place more to the *left* than the first product; and if there were any figures in the places for hundreds, thousands, &c. of the multiplier, I should place the corresponding products two, three, &c. places to the left. The whole of the work will now be intelligible.

The above row of products may be written down in any order we please; provided that, in commencing the multiplication by any other figure than the one in the units' place, we use proper caution in placing the product in its proper situation with respect to the decimal point.

Exs. 27. Express as simple decimals

1. $1·5 \times 1·5$.
2. $2·375 \times 3·48$.
3. $·006 \times 78·928$.
4. $1·0006 \times 461·8$.
5. $10·375 \times ·0074$.
6. $1·5 \times 3·15 \times 2·17$.
7. $·008 \times 5·5 \times 1·4$.
8. $3·75 \times ·014 \times ·875$.
9. $3·125 \times 14·25 \times ·01$.
10. $35·01 \times 7·98 \times 1·0001$.

118. If it be required that the product of two decimals be correct only to a certain number of decimal places, the above work may be contracted. For instance, let it be required to find the product of 7·24651 and 81·4632, correct to 4 places of decimals. First working the Ex. at full length, we have

```
   7·24651
  81·4632
     A
    14|49302
   217|3953
  4347|906
 28986|04
 72465|1
5797208|
-------|-----
590·3238|93432
```

By drawing a vertical line between the 5th and 6th columns, I cut off to the left that part of the product which will furnish the required 4 decimal places.

Now it must be observed that the column marked (A) is formed of the following products: viz. 7 in the upper line by 2 in the lower; 2 in the upper by 3 in the lower; 4 in the upper by 6 in the lower; 6 in the upper by 4 in the lower; 5 in the upper by 1 in the lower; 1 in the upper by 8 in the lower: but in forming this column alone, we must allow for the figures which ought to be carried from the columns not worked: and it will be found in practice that we shall be correct if, when our product is

between 5 and 15, we carry 1,
" 15 " 25, " 2,
" 25 " 35, " 3,
and so on.

This, however, is not to be done in multiplying by the last multiplier, as 8, when it multiplies the whole of the multiplicand, because in that product there can be no figure to the right from which to carry.

But this cross-multiplication is inconvenient, and may be avoided by the following arrangement.

Place the figure in the unit's place of the multiplier under that figure of the multiplicand whose place is the last to be retained in the product; and let the whole multiplier be arranged in precisely the reverse order.

We may now observe that every figure in the multiplier is under that figure in the multiplicand, by which, as was just now shown, it ought to be multiplied. And we shall produce the same work as before, by rejecting all the figures in the multiplicand that are to the right of any multiplying figure, setting down the products so that their right-hand figures may be in a vertical line, and carrying according to the directions given above.

The above Ex. will then be as follows:

```
  7·24651
  2364·18
 --------
 5797208
   72466
   28986
    4346
     217
      15
 --------
590·3238
========
```

I here place the 1 in the unit's place under the *fourth* decimal place 5, because I had to retain *four* decimal places in the product.

In the first multiplication, by 8, 1 carried nothing; in the second I carried 1; in the third 2; in the fourth 2; in the fifth 1; in the sixth 1; and upon adding up I find my result to be 590·3238, as before.

Ex. IV. Find the vaiue of 435·789 × 31·27, having no decimals in tne product. Working the Ex. both ways, I have

```
     435·789              435·789
       31·27              721·3
   -----------           --------
     30|50523             13073
     87|1578                436
    435|789                  87
  13073|67                   31
   -----------           --------
  13627·12203             13627
  ===========            ========
```

Exs 28. Find the value of

I. 3·275 × ·0175; 13·458 × 271·36; each correct to 2 places.

II. 135·849 × 1·0758; 458·19 × ·0375; each correct to 4 places.

III. 17·58 × 1·375; 1·5 × 1·5 × 1·75; each correct to whole numbers.

119. Next, let the decimals to be multiplied together be either one or both of them circulating. These may be converted into equivalent fractions; and then, after having multiplied them, we may convert the product into a circulating decimal; or the product of the decimals themselves may be found by Simple Multiplication, provided that there be taken a sufficient number of figures to enable us to ascertain the period in the product. We will work two Exs. to illustrate both methods

Ex. V. Find the value of $\cdot\dot{3}\dot{6} \times 75$.

By the first method I have

$$\cdot\dot{3}\dot{6} \times 75 = \frac{\overset{4}{\cancel{36}}}{\underset{11}{\cancel{99}}} \times 75 = \frac{300}{11} = 27\cdot2727 \ldots\ldots\ldots = 27\cdot\dot{2}\dot{7}.$$

By the second method, I write down the period three times, so that there may be figures enough in the product to show its period.

If a fourth period had been written in the multiplicand, the figures carried from the multiplication of it would have caused the period 27 to have been seen in the fifth and sixth decimal places in the product: and every additional period in the multiplicand would have produced one more period in the product: hence, since the number of periods, 36, in the multiplicand is unlimited, so, also, will be the number of periods, 27, in the product; *i. e.* the product is $27\cdot\dot{2}\dot{7}$, as before.

$$\begin{array}{r} \cdot 363636\ldots \\ 75 \\ \hline 1818180 \\ 2545452 \\ \hline 27\cdot272700 \end{array}$$

Ex. VI. Find the value of $37\cdot\dot{3} \times 9\cdot 1\dot{6}$.

Upon working this Ex. by ordinary Multiplication. I find that even if I write down five periods in the multiplicand, and four in the multiplier, yet the product is such that the learner would hardly detect the period in it; and if the periods had contained several figures, the work would be exceedingly heavy; and since in such Exs. involving two or more recurring decimals, the former method is the better, I give it alone. I then have

$$37\cdot\dot{3} \times 9\cdot 1\dot{6} = 37\tfrac{3}{9} \times 9\tfrac{15}{90} \qquad \text{by (110)}$$

$$= 37\tfrac{1}{3} \times 9\tfrac{1}{6}$$

$$= \frac{\overset{56}{\cancel{112}}}{\cancel{3}} \times \frac{55}{\underset{3}{\cancel{6}}}$$

$$= \frac{3080}{9} = 342\cdot\dot{2}.$$

Exs. 29. Form the following products.

1. $\cdot\dot{3} \times 1\cdot 78$.
2. $1\cdot\dot{2}\dot{7} \times \cdot 0458$.
3. $13\cdot 0\dot{0}7\dot{9} \times 4\cdot 5$.
4. $27\cdot\dot{3} \times 4\cdot\dot{6}$.
5. $1\cdot\dot{6} \times \cdot\dot{3}48\dot{5}$.
6. $3\cdot 58 \times \cdot\dot{2}\dot{1} \times 4\cdot 1\dot{6}$.

DIVISION.

(Thrower, Case VIII.)

120. In the Exs. under this head, either dividend or divisor, or both, may be a decimal. I will give one Ex. of each variety.

Ex. I. Find the value of $37\cdot5 \div 84$.

Where, as in this Ex., the divisor is a composite number containing no prime factor greater than 12, we can divide by successive factors, as in (100); thus:

$$\frac{37\cdot5}{84} = \left(\frac{37\cdot5}{12\times7} = \right)\frac{3\cdot125}{7} = \cdot44642857142, \&c.$$
$$= \cdot446\dot{4}2857\dot{1}.$$

121. Also, since we have shown that by moving the point in any decimal from left to right, we in reality multiply that decimal by a power of 10; therefore if we have a decimal quantity as divisor, we can write the sum as a fraction $\left(\frac{\text{dividend}}{\text{divisor}}\right)$, and remove the point so many places to the right in both divisor and dividend as will cause the divisor to be a whole number. By this process we shall merely have multiplied both num^{r} and den^{r} of a fr^{n} by the same power of 10. The remaining part of the division will be merely an Ex. such as in (97) to (102); and the position of the point will be determined as in that case.

Ex. II. Find the value of $716\cdot343069 \div 27\cdot69$.

$$\text{The quotient} = \frac{716\cdot343069}{27\cdot69} = \frac{71634\cdot3069}{2769} = 25\cdot8701. \qquad \text{(S)}$$

and the correctness of this position of the point may be shown immediately: for, throwing 27·69, the den^{r} of the first fraction, into the numerator of the right-hand side, I have,

$$716\cdot343069 = 25\cdot8701 \times 27\cdot69;$$

where I observe, that if I had to form the product expressed in the right side of the equation, I should have 6 decimal places in it, as I see to be the case in the left-hand side: hence it is clear that the point was rightly placed in the quotient 25·8701. And further, observing the left hand fraction in (S), I see that by subtracting the number of decimal places in the divisor from that in the dividend, I obtain the number of places in the quotient. Hence we take it as a Rule, that the division should be performed as in whole numbers, and that there should be pointed off in the quotient as many decimal places, as the number in the dividend exceeds the number in the divisor.

Assuming this Rule, we will now work an Ex. as in Long Division of integers, and point off the proper number of decimal places, as soon as all the figures in the dividend have been brought down.

Ex. III. Find the value of 41·0632884 ÷ ·0438.

```
·0438) 41·0632884 (937·518
       3942
       ----
        1643
        1314
        ----
         3292
         3066
         ----
          2268
          2190
          ----
            788
            438
            ----
            3504
            3504
            ----
```

Here I work precisely as though the divisor and dividend were integers; and when I have completed the division, I observe that as there are 7 places in the dividend, and 4 in the divisor, there must therefore be 3 places in the quotient.

122. If, however, the number of places in the dividend be less than that in the divisor, there must be appended as many ciphers to the dividend as shall make up the number of places in the dividend at least equal to the number of places in the divisor. If the dividend be a whole number, a point must be placed after the units' place, and then ciphers may be appended: and it is clear that these ciphers will not in any way alter the value of the dividend.

Ex. IV. 971·7 ÷ ·123.

Here appending ciphers and working as in Simple Long Division, I have

```
·123) 971·700 (7900
      861
      ----
      1107
      1107
      ----
        00
```

and since the number of decimal places in the divisor and dividend is the same, therefore the number of places in the quotient will be 0; or the quotient will be an integer.

Sometimes, also, though there be a sufficient number of decimal places in the dividend to give one or two places in the quotient, yet if it is required that there should be 5 or 6 decimal places in the quotient, ciphers may be appended to the dividend, as before, and the division continued as far as we please.

Ex. V. 62·5 ÷ ·025.

This Ex. I work in two ways; first, by removing the point three places to the right in the numerator and denominator, as in (96); and secondly, by converting the given decimals into fractions, and performing the division by the usual method.

$$\frac{62{\cdot}5}{{\cdot}025} = \frac{62500}{25} = \frac{12500}{5}$$

$$= 2500$$

$$\text{or } \frac{62{\cdot}5}{{\cdot}025} = \frac{625}{10} \div \frac{25}{1000} = \frac{\overset{25}{\cancel{625}}}{\cancel{10}} \times \frac{\overset{100}{\cancel{1000}}}{\cancel{25}}$$

$$= 2500, \text{ as before.}$$

This second operation shows by an independent method that the Rule which we have laid down for pointing is correct; and though it is not convenient for general use, yet a pupil will do well to try it upon a few simple examples.

Exs. 30. Find the required quotients in the following examples:—

1. 13·5 ÷ ·15.	4. 345·6 ÷ 1·728.	7. 576·84325 ÷ 119·3.
2. 83·75 ÷ ·128.	5. 13·358697 ÷ ·634.	8. 3·84765 ÷ 1536.
3. 1080 ÷ .008.	6. ·084007 ÷ 34·3.	9. 1·0005 ÷ 106·3.

As in Mult^{n}, so also in Div^{n}, a contracted form of working may be employed, where it is intended to retain only a limited number of places in the quotient. The following is the Rule. Take as many of the left-hand figures in the divisor as will make up the *entire* number of figures required in the quotient. Divide by the divisor so chosen, and at every succeeding division drop one figure at the right-hand of this divisor, carrying to the right-hand figure of each product, according to the scale used in contracted multiplication.

Obs. When there are not as many figures in the divisor as are required in the quotient, proceed with the division, until the divisor contains as many figures as *remain* to be found in the quotient: then commence the contracted division upon the same plan as before.

In the following example worked both ways, the principle of contraction will be seen to be the same as that in which one or more ciphers are cut off from divisor and dividend.

Ex. Divide 327·856 by 45·62, leaving 3 places of decimals.

```
45·62) 327·856 (7·186          45·62) 327·856 (7·186
       31934                          31934
       -----                          -----
        8516                            851
        4562                            457
       -----                          -----
        39540                           394
        36496                           364
        -----                          -----
         30440                           30
         27372                           26
         -----                          -----
          3068                            4
```

It is here plain that 327 ÷ 45 will give one integer; so that the quotient, when containing 3 places of decimals will in all contain 4 figures; I therefore retain the whole divisor in the first division, and then proceed according to my Rule, cutting off one figure from the divisor, and carrying the numbers 1, 4, and 2, in forming the successive subtrahends.

Exs. 31. Find the value of

1. 89·7643 ÷ 15·827 to 3 places.
2. 126·4906 ÷ 3·274 to 4 places.
3. 45·87623 ÷ 12·897 to 2 places.
4. 36·045 ÷ 2·75, retaining only integers in the quotient.

123. If either or both of the given decimals circulate, the circulator may be converted into a proper or improper frⁿ, as the case may be, and the division proceeded with as in fractions; or the circulator may be left unaltered, if in the dividend, and only the divisor be converted into a frⁿ, and the division then performed.

Ex. I. Find the value of $75 \div \dot{1}4\dot{8}$.

By (108) $$\cdot\dot{1}4\dot{8} = \frac{148}{999} = \frac{4}{27} \text{ (in lowest terms)}$$

therefore $$\frac{75}{\cdot\dot{1}4\dot{8}} = \frac{75}{\frac{4}{27}} = \frac{75 \times 27}{4} = \frac{2025}{4} = 506\cdot25.$$

Ex. II. Find the quotient of $\cdot 96\dot{3}4\dot{5}$ when divided by $\dot{3}$.

$$\cdot\dot{3} = \frac{3}{9} = \frac{1}{3}$$

therefore $$\frac{\cdot 96\dot{3}4\dot{5}}{\cdot\dot{3}} = \frac{\cdot 96\dot{3}4\dot{5}}{\frac{1}{3}} = (\cdot 963453345\ldots\ldots) \times 3$$

$$= 2\cdot 89036035\ldots\ldots$$

$$= 2\cdot 890\dot{3}\dot{6}.$$

It has been stated in (113) and (122) that ciphers may be written after the last figure of a decimal without altering its value: similarly, they may be cut off without affecting it; and this is generally done, if the decimal resulting in any Ex. have ciphers at the end. Thus in Art. 116, Ex. II., I might have removed the cipher which is at the end of the final decimal; only that a pupil would have thought that there were pointed off only 11 decimal places, instead of 12: but, by referring to the last vulgar fraction used in that Ex., I find that it might have been reduced to lower terms, by dividing numerator and denominator by 10; and there would then have been but 11 ciphers in the den^{r}, and consequently 11 places in the decimal which follows.

Exs. 32. Find the required quotients in the following Examples:—

1. $\cdot\dot{3}\dot{6} \div \cdot072$.	4. $1\cdot23123\ldots\ldots \div \dot{3}\cdot\dot{6}$.
2. $27\cdot\dot{5} \div \cdot0\dot{6}$.	5. $\cdot18 \times \cdot09 \div \cdot1\dot{6}$.
3. $34\cdot7\dot{5} \div 1\cdot\dot{5}\dot{6}$.	6. $13\cdot\dot{7}\dot{5} \div (1\cdot\dot{3} + 5\cdot\dot{6})$.

REDUCTION OF DECIMALS.

(Thrower, Cases IX. and X.)

124. We have here to perform in Decimals the operation which in (62) was performed in Vulgar Fractions; and this merely requires that the decimal quantity should be reduced to successive lower denominations, until we have either no decimal part remaining, or until we reach the lowest denomination used.

Ex. I. Express in positive terms ·375 of £1.

·375 of £1 = ·375 × 20s. = 7·500sh.;
and ·5sh. = ·5 × 12d. = 6·0d. = 6d.;
therefore ·375 of £1 = 7s. 6d.

$$\begin{array}{l} \pounds \\ \cdot 375 \\ \underline{\quad 20} \\ 7\cdot 5\cancel{00}\text{sh.} \\ \underline{\quad 12} \\ 6\cdot 0\text{d.} \end{array}$$

The work may be written out as annexed: where it is plain that at the end of each successive multiplication I point off as many decimal places as there were in the preceding line, because there are no decimal places in the multiplier.

It will be seen that in both the above operations I might have omitted two ciphers at the end of the decimal part in the shillings, for the reason given at the close of the last article. I have therefore crossed them out.

By using the equalities mentioned at the end of (101), and referring to the aliquot parts of different denominations given in (64), we might have worked this Ex. very briefly.

Thus, $\cdot 375$ of £1 $= \frac{3}{8}$ of £1 = 7s. 6d.

So, also, $\cdot 875$ of £1 $= \frac{7}{8}$ of £1 = 17s. 6d.

Ex. II. Find the value of 7·14685 of 5s. 6¾d.

Here, since the concrete number is expressed in several denominations, I must either reduce the decimal to a fraction, or reduce the 5s. 6¾d. to the fraction of a penny, and then perform the multiplication.

By the second method:

$$\begin{aligned} 7\cdot 14685 \times 5\text{s. } 6\tfrac{3}{4}\text{d.} &= 7\cdot 14685 \times 66\tfrac{3}{4}\text{d.} \\ &= 7\cdot 14685 \times 66\cdot 75\text{d.} \\ &= 477\cdot 0522375\text{d.} \\ &= 39\text{s. } 9\cdot 0522375\text{d.} \\ \textit{Answer.} &= \pounds 1\ 19\text{s. } 9\cdot 0522375\text{d.} \end{aligned}$$

Just as in (63) I left the remainder as a fractional part of a penny, so here I leave it as a decimal fraction of a penny.

125. If any circulating decimals occur in Exs. under this head, we must reduce them to fractions, and proceed as before.

Ex. III. Find the value of $\cdot\dot{5}$ of £1 11s. 4d.

$$\begin{aligned} \cdot\dot{5} \text{ of } \pounds 1\ 11\text{s. } 4\text{d.} &= \frac{5}{9} \text{ of } 31\tfrac{1}{3}\text{sh.} \\ &= \frac{5}{9} \times \frac{94}{3}\text{sh.} = \frac{470}{9 \times 3}\text{sh.} = \frac{156\cdot 666\ldots\ldots}{9}\text{sh.} \\ &= 17\cdot 4074074\ldots\ldots\text{sh.} \end{aligned}$$

$$\begin{aligned} \text{and } \cdot 407407\ldots\ldots\text{sh.} &= (\cdot 407407\ldots\ldots) \times 12\text{d.} \\ &= 4\cdot 888884\ldots\ldots\text{d.} \\ &= 4\cdot\dot{8}\text{d.} \\ &= 4\tfrac{8}{9}\text{d.} \end{aligned}$$

therefore $\cdot\dot{5}$ of £1 11s. 4d. = 17s. $4\frac{8}{9}$d.

Exs. 33. Express in positive terms

1. $1 \cdot 375$ of 1 guinea.
2. $\cdot 028$ of a moidore.
3. $375 \cdot 794$ of £5.
4. $1 \cdot 115$ of a crown.
5. $\cdot 148325$ of 7s. 6d.
6. $49 \cdot 864$ of £15 10s.
7. $\cdot 05$ of $7\frac{1}{2}$d. + $\cdot 375$ of 3s. 6d.
8. $11 \cdot 05$ of £2 − $17 \cdot 5$ of 6s. 8d.
9. $\cdot 375$ of a mile + $7 \cdot 5$ of a yard.
10. $1 \cdot 185$ of a cwt. − $\cdot 0375$ of a qr.
11. $\cdot \dot{5}$ of 6s. 8d. + $1 \cdot \dot{2}\dot{7}$ of a guinea.
12. $\cdot \dot{0}3\dot{7}$ of 27s. − $\cdot 02$ of £2.

126. The following operation is the converse of that performed in the last two Arts., and corresponds to that exhibited in (78) in Vulgar Fractions.

Ex. Reduce 4s. $6\frac{1}{2}$d. to the decimal of a sovereign.

In working this Ex. we shall first reduce the former of the given quantities to the fraction of the latter, and then convert into a decimal the vulgar fraction connecting the two quantities.

$$\frac{4\text{s. } 6\frac{1}{2}\text{d.}}{1£} = \frac{54\frac{1}{2}\text{d.}}{240\text{d.}} = \frac{54 \cdot 5}{240} = \frac{4 \cdot 54166\ldots.}{20} = \frac{2 \cdot 27083\dot{3}}{10} = \cdot 227083\dot{3}$$

or 4s. $6\frac{1}{2}$d. = $\cdot 227083\dot{3}$ of £1.

Exs. 34.

1.	Reduce	2s. 6d......	to the decimal	fraction of	15s.
2.	„	3s. 7d......	„	„	£5.
3.	„	25s........	„	„	3 guineas.
4.	„	£17 15s....	„	„	£100.
5.	„	354 yds.....	„	„	1 league.
6.	„	11440 yds...	„	„	2 acres.
7.	„	$3\frac{1}{2}$ qrs......	„	„	15 tons.
8.	„	6 hours	„	„	135 days.
9.	„	1 leap year..	„	„	3 weeks 4 days.
10.	„	$\frac{3}{5}$ of a mark..	„	„	$\frac{9}{11}$ of a crown.

MISCELLANEOUS EXAMPLES.

127. The same remark applies to these Miscellaneous Examples in Decimals that applied to the corresponding Exs. in Fractions: and the only general assistance that can be given to a pupil is to show him the neatest method of performing the operations required.

Ex. I. Multiply £8 17s. 6d. by 75·25, and reduce the result to the decimal of £100.

$$\text{Here } \frac{(\pounds 8\ 17\text{s.}\ 6\text{d.}) \times (75{\cdot}25)}{\pounds 100} = \frac{\pounds 8\tfrac{7}{8} \times 75{\cdot}25}{\pounds 100}$$

$$= \frac{8{\cdot}875 \times 75{\cdot}25}{100}$$

$$= \frac{667{\cdot}84375}{100}$$

$$= 6{\cdot}6784375$$

Ex. II. Reduce $\frac{3{\cdot}275}{405}$ of $\frac{2{\cdot}5}{{\cdot}075} \times \frac{3{\cdot}125}{11} \times \frac{9}{9{\cdot}375}$ to a simple quantity.

Moving the point three places to the right in two numerators and in two denominators, and again one place to the right in the numerator and the denominator of the second fraction, the expression becomes

$$\frac{\overset{\overset{131}{\cancel{655}}}{\cancel{3275}}}{\underset{\underset{9}{\cancel{81}}}{\cancel{405}}} \times \frac{\cancel{25}}{\underset{\underset{6}{\cancel{30}}}{\cancel{750}}} \times \frac{\cancel{3125}}{11} \times \frac{\cancel{9}}{\underset{3}{\cancel{9375}}} = \frac{131}{9 \times 6 \times 11 \times 3}$$

$$= \frac{11{\cdot}90909\ldots\ldots}{9 \times 6 \times 3}$$

$$= \frac{1{\cdot}323232\ldots\ldots}{6 \times 3}$$

$$= \frac{{\cdot}2205387205387 2\ldots\ldots}{3}$$

$$= {\cdot}07351290684624\ldots\ldots, \text{ \&c.}$$

Since a very large number of decimals is non-terminating, it might seem that vulgar fractions, which are always expressed in finite terms, would be preferable for every purpose. But this is not the case, for decimals have one advantage over fractions from the following consideration.

In ascertaining the comparative value of two or more fractional quantities, if they be expressed as vulgar fractions, it is necessary to reduce them to a common denominator; but if they are represented as decimals, mere inspection will detect their comparative value, as readily as can be done in whole numbers.

For example, if we have to compare $\frac{7}{15}$, $\frac{11}{25}$, $\frac{13}{28}$, we cannot see which is the largest, and which the least, without reducing them to a common denominator: but if the quantities had been written in their decimal form, viz.

$$\cdot4666\ldots\ldots\ldots; \quad \cdot44; \quad \cdot46\dot{4}2857\dot{1},$$

we could see *at once* that the first is the largest; the last one is the next; and the middle one is the smallest: therefore, in order of magnitude they are $\frac{7}{15}$, $\frac{13}{28}$, $\frac{11}{25}$. The decimal form is more especially useful, when several fractional quantities are arranged in a table, and where it is requisite to be able to compare the different quantities at a glance.

We may take as an Ex. the accompanying table, in which the numbers represent the comparative weights of equal bulks of different substances.

Sheet Glass	3·33
Plate Glass	2·5
Marble	2·716
Quartz	2·6
Rock Salt..........	1·92
Ivory	1·917
Ice at 0°	·926
Water at 60°	1·

Though the 5th and 6th numbers are very nearly equal, yet it can be seen at once that the 5th is larger than the 6th by three thousandths: as fractions, these quantities would have been written, $1\frac{23}{25}$; and $1\frac{917}{1000}$; and the difference could not have been ascertained by inspection.

Exs. 35.

1. Express as a simple decimal the difference between $\frac{3}{5}$ of $\frac{11}{12}$ and $\frac{7}{8}$ of $\frac{9}{14}$.

2. Simplify the following expression:

$$\frac{1\cdot5}{\cdot075} \times \frac{3\cdot25}{1\frac{1}{8}} + \frac{1\cdot875}{2\cdot1} \times \frac{3\cdot5}{3\cdot75}.$$

3. What decimal of a groat is equivalent to a crown?

4. Form the following quantities into a decimal table as above, arranging them in order of magnitude;

$$2\frac{7}{8}, \quad 1\frac{3}{4} \text{ of } 1\cdot06, \quad \frac{2}{7\cdot5}, \quad 1\cdot75 \text{ of } 3.$$

5. What is the average value of the above quantities expressed as a vulgar fraction?

6. Find the simple decimal equivalent to $\dot{1}\cdot\dot{3} \times (2\cdot\dot{4} + 7\cdot\dot{5})$.

7. Express in positive terms the sum of 1·05 of a crown, $\cdot\dot{7}1428\dot{5}$ of a guinea, and $\cdot 1\dot{6}$ of 6s. 8d.

8. Convert $\frac{5}{16}$ into a decimal, by multiplying both num^r and den^r by some common quantity.

9. Reduce $2\frac{3}{8}$ of $14\frac{1}{2}$ acres to the decimal of a square mile.

10. Find in a decimal form a fourth proportional to each of the following sets of numbers:

$$1,\ \cdot 2,\ \cdot 375; \qquad 3\cdot 25,\ \cdot 0175,\ 1\cdot 01; \qquad \frac{1\frac{1}{2}}{3},\ \frac{3}{7},\ 2\cdot 75.$$

Exs. 36. E.

1. The product is 154923000, and multiplicand 42375; what is the multiplier?

2. How many calendar months in $3\frac{1}{4}$ centuries?

3. If 90 degrees = 100 grades, find the number of degrees in $12\frac{1}{2}$ grades.

4. A man enters into business with £20,000, and each year makes a profit of one-fourth of his investment, and adds that profit to his capital; how much will he be worth in 5 yrs.?

5. How many fathoms are there in a degree?

6. If there are 360° in every circle, and in latitude 30° a degree = 34·75 miles, what is the length of the circle which passes through latitude 30°?

7. What is the average length of the calendar months, including leap year?

8. Find the abstract number which expresses the ratio of £$12\frac{3}{4}$ and $17\frac{3}{8}$ shillings.

9. Assuming that the number of square feet in the area of an oblong surface is found by multiplying the number of feet in the length by the number in the breadth, find how many bricks, each 9 in. by $4\frac{1}{2}$ in., are required to pave a floor $97\frac{1}{2}$ ft. long, and 81 ft. broad.

10. Write an equation involving the signs of add^n, sub^n, mult^n, and div^n, and having the right-hand side = 237.

11. Explain the object and the process of reducing fractions to a C. D. Reduce to L. C. D. $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{9}$, explaining the work.

12. Convert into a vulgar fraction, in lowest terms, the sum of 9 tenths, 3 hundredths, and 5 thousandths.

13. Multiply the difference between 75 hundredths and 5 tenths, by 8 thousandths; expressing the result as a vulgar fraction.

14. Reduce the following expression to a simple decimal:—

$$\frac{1}{2}\left(6\frac{1}{2} + 2\frac{2}{3} - 3\right).$$

15. What decimal of £1 is $\frac{2}{5}$ of 13s. 4d.?

F.

1. There are 100 links in a chain of 4 perches; how many links in a league?

2. The wheel of a locomotive which is $16\frac{1}{2}$ ft. in circumference, turns round 25764 times, how many miles will it have run?

3. Find the cost of half an acre of building land, at 3s. 6d. per sq. yd.

4. What is the expense of painting the walls of a room, 17 feet long, 13 ft. broad, and 10 ft. high, at 7d. a square yard?

5. On a railway, from *A* to *B* there is a rise of 1 in 160 for $\frac{1}{2}$ mile, of 1 in 330 for 2 miles 50 yards, then a fall of 1 in 110 for 1 mile 360 yards, lastly, a rise of $\frac{1}{220}$ for 1500 yards; what is the height of *A* compared with *B*?

6. A general sends away $\frac{13}{28}$ of his army, and then $\frac{2}{5}$ of the remainder; he has now 1350 men, what had he at first?

7. Find the exact value of $\cdot 1\dot{6}$ of a moidore + $\cdot\dot{5}7142\dot{8}$ of 2 guineas.

8. How many allotments, each 2r. 25p., are contained in $47\frac{1}{2}$ acres?

9. If there are 100 links in a chain of 22 yards, find how many square links are contained in an acre.

10. A bankrupt owes £3840, and his property amounts to only £1656; how much will his creditors receive in the pound?

11. If a board be $23\frac{1}{2}$ inches broad, how long must it be to contain 15 square feet?

12. If 500 slates would cover a surface 25 feet square, how many would be required for a roof 27 ft. by 36?

13. What number divided by $17\frac{1}{6}$ will give $13\frac{7}{8}$?

14. To $\frac{9}{10}$ of a gross add $\frac{3}{5}$ of a quarter of a hundred; from this sum subtract $\frac{7}{8}$ of a score, and divide the remainder by $17\frac{1}{6}$.

15. Simplify the expression $\frac{3}{5}$ of $\frac{10}{27}$, proving that the mode of working is correct; and explain what is called *cancelling*.

G.

1. If the interest of the national debt be £1 2s. 6d. each second of time; what is the amount per annum?

2. How many bottles in 103 casks, each containing $9\frac{3}{4}$ dozen?

3. A person pays a debt of £230 8s. in sovereigns, half sovereigns, crowns and shillings, of each an equal number; how many of each?

4. What is the amount of the following items, $54\frac{1}{4}$ lbs. at 5s. $4\frac{1}{2}$d. 241 yds. at $7\frac{1}{2}$d., and 512 pieces at 2s. $4\frac{3}{4}$d.?

5. Express as a Vulgar Fraction, in its lowest terms, the sum of 7 hundredths, 5 thousandths, and 375 tenths of thousandths.

6. How much paper $1\frac{1}{4}$ yards broad, will cover as much as 20 yards, of $\frac{3}{4}$ yd. broad?

7. If 4 lb. of silver be mixed with 4 lb. 5 oz. of gold, how much silver will there be to 6 oz. of gold?

8. If one whose rent is £430 pays a tax of £30 6s. 9d., what should be the rent of a man whose taxes come to £94 16s. $1\frac{1}{4}$d.?

9. Gunpowder being composed of nitre 15 parts, charcoal 3 parts, and sulphur 2 parts; find how much of each is required in making 16 cwt. of powder.

10. I take successively $\frac{1}{3}$ and then $\frac{1}{5}$ of a sum of money, and find that I have left £15; what was the sum at first?

11. Explain the two methods of multiplying a fraction by a whole number, taking as an Ex. $\frac{5}{6} \times 3$.

12. In a bridge of 7 arches, the middle one is 75 feet span, and the others on each side are $\frac{1}{10}$th less in each succeeding arch; find the whole length of the bridge, allowing 15 feet for each pier.

13. Find the value of $\left(\frac{4}{5\frac{1}{3}} \text{ of } £5\right) \sim 3{\cdot}\dot{4}65\dot{3}$ of a guinea.

14. What is the price per lb. of an article, of which 1·5 cwts. cost 113·75 shillings?

15. Express in positive terms $\cdot 37\dot{6}$ of 27s. 6d. $\sim 1{\cdot}\dot{8}$ of 2 guineas.

H.

1. How many bottles of wine in 12 pipes, at the rate of 52 dozen 9 bottles each pipe?

2. Find the number of square yards in an area, $\frac{1}{2}$ a mile long, and $\frac{1}{4}$ of a mile broad.

3. If one man thrash 7 sheaves of corn in a day, and each sheaf yield $3\frac{1}{4}$ pecks, and each peck 15 lbs.; how much in quantity and weight will 15 men thrash in 6 weeks?

4. A general having an army of 24000 men, increases it one-third by recruiting; afterwards he loses one-fourth by disease, and of the remainder one-fifth fell in battle; how many men has he left?

5. How many fathoms in a degree?

6. What is the ratio of a geographical mile to a British mile? How many geographical miles must I measure, so as to contain the least exact number of British miles?

7. If a person step at an average $2{\cdot}1\dot{6}$ feet, how many steps must he take in ·325 miles?

8. Reduce $\frac{7}{256}$ to a decimal without using Long Division; and shew that $\frac{3}{125} = {\cdot}024$, without dividing at all.

9. Find the value of $4{\cdot}25 \div {\cdot}10625$, proving the result by Vulgar Fractions.

10. Write in words ·75, 2·0324, 17·000001.

11. Find the value of (£25 16s. $7\frac{5}{8}$d.) $\times 85\frac{11}{12}$.

12. The circumferences of the fore and hind wheels of a carriage are respectively $9\frac{3}{7}$ ft. and $13\frac{3}{11}$ ft.; find how many more revolutions one makes than the other in $10\frac{3}{8}$ miles.

13. What fraction of a guinea and a half, together with £3 16s. 9d., will give £4?

14. Exhibit as a simple vulgar fraction the result of

$$\cdot3\dot{7} \div 1\cdot17 \times \cdot052 \div 1\cdot\dot{8}.$$

15. Find the ratio between the product and quotient of 147 and $\cdot\dot{2}\dot{7}$.

PRACTICE.

128. Practice is a Rule which endeavours to show the readiest method of finding the cost of any number of articles at a certain price: and the work exhibited in any Ex. consists of a series of amounts such as a person would try to obtain, if he were working the question mentally.

Thus, if I had to find the value of 54lbs. at $1\frac{1}{2}$d. each, I should say, 54 at 1d. = 54d. = 4s. 6d.; and 54 at $\frac{1}{2}$d. = 54 halfpence = 27d. = 2s. 3d.: and therefore 54 at $1\frac{1}{2}$d. = 4s. 6d. + 2s. 3d. = 6s. 9d.

129. But if the number of articles had been much larger, or the price much greater, the value of them could not readily have been obtained mentally: we therefore, in Practice, use the above method of mental calculation, but we write down the successive results, and find their sum for the final result.

The Exs. to be worked under this Rule may be arranged as follows:—

When the price is under a shilling, as

Ex. I. 4108 at $7\frac{3}{4}$d.

When the price is between 1s. and £1, as

Ex. II. 4103 at 7s. $5\frac{1}{2}$d.

Ex. III. 6009 at 19s. $5\frac{1}{4}$d.

When the price consists of more than £1, as

Ex. IV. 7111 at £1 17s. $4\frac{1}{2}$d.

Ex. V. 4013 at £12 7s. $0\frac{1}{4}$d.

When there is a fraction in the given number of quantities, as

Ex. VI. $6583\frac{4}{15}$ at £1 19s. $11\frac{3}{4}$d.

When there is a fractional part of a penny other than farthings, as

Ex. VII. 4176 at £3 5s. $4\frac{8}{10}$d.

When the quantity, the price of which is required, consists of several denominations, as

Ex. VIII. 9lb. 3oz. 14dwt. at £10 15s. 6d. per lb.

130. Mention was made in (64) of certain fractional parts of £1, 1s. &c., which were termed aliquot parts of £1, 1s., &c. It is advisable to have such parts of the denominations most in use familiarly in the mind; but a pupil will find in PRACTICE, that he has to take aliquot parts of many other quantities which are intermediate between such standard units as £1, 1cwt., &c.: and nothing will render him expert in taking such aliquot parts as he will require, but a readiness in the treatment of fractions.

The following are the most useful aliquot parts of £1 and 1s., and should therefore be remembered.

10s. 0d.	$= \frac{1}{2}$ £
6s. 8d.	$= \frac{1}{3}$ „
5s. 0d.	$= \frac{1}{4}$ „
4s. 0d.	$= \frac{1}{5}$ „
3s. 4d.	$= \frac{1}{6}$ „
2s. 6d.	$= \frac{1}{8}$ „
1s. 8d.	$= \frac{1}{12}$ „
1s. 4d.	$= \frac{1}{15}$ „
1s. 3d.	$= \frac{1}{16}$ „
1s. 0d.	$= \frac{1}{20}$ „

6d.	$= \frac{1}{2}$ shilling.
4d.	$= \frac{1}{3}$ „
3d.	$= \frac{1}{4}$ „
2d.	$= \frac{1}{6}$ „
$1\frac{1}{2}$d.	$= \frac{1}{8}$ „
1d.	$= \frac{1}{12}$ „

In all cases we shall find, that where the price given is not an aliquot part of the unit of the next higher denomination, it is necessary to split the price into two or more portions, of which the largest must be an aliquot part of this said unit, and the remaining portions are aliquot parts either of this same unit, or of some one of the portions already used in the Ex.

Ex. I. 4108 at $7\frac{3}{4}$d.

In this Ex. since the price $7\frac{3}{4}$d. is not an aliquot part of 1s., I therefore break up this price into three parts, 6d., $1\frac{1}{2}$d, and $\frac{1}{4}$d., of which the largest, 6d. = $\frac{1}{2}$ of 1s. the next higher denomination; the next, $1\frac{1}{2}$d. is an aliquot part of 6d., viz. $\frac{1}{4}$th; and the last, $\frac{1}{4}$d. is $\frac{1}{6}$th of $1\frac{1}{2}$d. Now, I know that 4108 articles at 1s. each would cost 4108s.; therefore, 4108 at 6d., *i. e.* at $\frac{1}{2}$sh., cost 4108 times $\frac{1}{2}$sh., or $\frac{1}{2}$ of 4108s.: I therefore find the value of this quantity and write it down, viz. 2054s. So also, 4108 at $1\frac{1}{2}$d. = 4108 at $\frac{1}{4}$ of 6d., and therefore = $\frac{1}{4}$th of the value of 4108 at 6d., which was found just before. Similarly, since $\frac{1}{4}$d. = $\frac{1}{6}$th of $1\frac{1}{2}$d., therefore 4108 at $\frac{1}{4}$d. = $\frac{1}{6}$th of the previously found value of 4108 at $1\frac{1}{2}$d. Hence the sum of my three amounts at 6d., $1\frac{1}{2}$d., and $\frac{1}{4}$d., will be the total value, at $7\frac{3}{4}$d. The whole of the operations which have just been described are to be written out in the accompanying shape.

Ex. I. 4108 at $7\frac{3}{4}$d.

			4108	
A price 6d.	= $\frac{1}{2}$sh.	gives	2054	
$1\frac{1}{2}$d.	= $\frac{1}{4}$ of 6d.	„	513	6
$\frac{1}{4}$d.	= $\frac{1}{6}$ of $1\frac{1}{2}$d.	„	85	7
$7\frac{3}{4}$d.		2,0	265,3	1
			£132 13	1

(See App. Art. Composite Divisor.)

Had the price been below 6d., as for instance 3175 at $2\frac{3}{4}$d., we should have taken as parts 2d. = $\frac{1}{6}$s.; $\frac{1}{2}$d. = $\frac{1}{4}$ of 2d.; and $\frac{1}{4}$d. = $\frac{1}{2}$ of $\frac{1}{2}$d.; and the remainder of the work as before.

In the second division in Ex. I. 4 being a divisor, I had to find the value of $\frac{2054\text{s.}}{4} = 513\frac{1}{2}\text{s.} = 513\text{s. } 6\text{d.}$; and generally, when as in (64) a number of shillings, or pounds is divided by a divisor, and a rem^r is left, the fractional quotient can be converted into positive terms *at once*, and more readily than by the usual method, in which the

remainder is reduced to lower den^n, and the division again performed. But in the next line, where the divisor is 6, we have to take a sixth part of the 6d. as well as of the 513s., hence I have $\frac{513}{6}$s. = $85\frac{5}{6}$s. = 85s. 6d.; and this, with the sixth part of 6d., viz. 1d., becomes 85s. 7d. Though it takes a long time to explain these processes, yet a pupil who is quick at working fractions will obtain the above results more readily than I can describe them, and much time will be saved by their use: but those who prefer the usual method of reducing the remainders, as in Compound Division, can of course adhere to it. The 2653 in the result evidently consists of shillings; and from this Ex. we see that the highest denomination in the sum of all the separate amounts is the same as that of which we took the first aliquot parts; and so also is the first remainder obtained in each of the divisions.

Exs. 37.

		d.			*d.*
1.	1857 at	$\frac{1}{4}$.	7.	6329 at	10.
2.	7151 „	$\frac{3}{4}$.	8.	8537 „	$8\frac{3}{4}$.
3.	3982 „	$1\frac{3}{4}$.	9.	11071 „	$10\frac{1}{2}$.
4.	6110 „	$5\frac{1}{4}$.	10.	7705 „	$10\frac{3}{4}$.
5.	3457 „	$9\frac{1}{4}$.	11.	3956 „	$11\frac{1}{2}$.
6.	7403 „	$9\frac{1}{4}$.	12.	8793 „	$11\frac{3}{4}$.

Ex. II. 4103 at 7s. $5\frac{1}{4}$d.

			4103		
A price 5s. 0d.	= $\frac{1}{4}$£	gives	1025	15	
2s. 0d.	= $\frac{1}{10}$£	„	410	6	
4d.	= $\frac{1}{6}$ of 2s.	„	68	7	8
1d.	= $\frac{1}{4}$ of 4d.	„	17	1	11
$\frac{1}{4}$d.	= $\frac{1}{4}$ of 1d.	„	4	5	$5\frac{3}{4}$
7s. $5\frac{1}{4}$d.			£1525	16	$0\frac{3}{4}$

In breaking the price 7s. $5\frac{1}{4}$d. into parts, we find that 2s. is $\frac{1}{10}$th of £1, and not of the previous aliquot part 5s.; hence, in dividing by 10, we must take as dividend, not the line corresponding to 5s. but the top line, 4103, which is the value of 4103 at £1 each: and in taking aliquot parts, we must always be careful to take as dividend, that line which expresses the value given by that coin or denomination of which we are taking an aliquot part.

Obs. In this and the preceding Ex. I have written the denomination of which the aliquot part has been taken in every line: but for the future I shall generally omit the denomination when I am taking an aliquot part of the line preceding, but insert it when I am taking a part of some earlier dividend.

Exs. 38.

			s.	d.
1.	7992	at	1	7½.
2.	8495	„	2	9¾.
3.	3708	„	3	7¼.
4.	8222	„	5	10¾.
5.	1007	„	7	11¼.
6.	1173	„	3	8¼.
7.	4351	„	8	0½.
8.	3672	„	7	10.

			s.	d.
9.	10205	„	4	10½.
10.	7248	„	9	1¼.
11.	12826	„	8	11¼.
12.	8793	„	9	7¾.
13.	1250	„	9	11½.
14.	7108	„	8	6¾.
15.	11489	„	9	10
16.	12146	„	9	8¾.

Ex. III. 6009 at 19s. 5¼d.

			6009		
A price 10s. 0d.	= ½£	gives	3004	10	
5s. 0d.	= ½	„	1502	5	
4s. 0d.	= ⅕£	„	1201	16	
4d.	= $\frac{1}{12}$	„	100	3	
1d.	= ¼	„	25	0	9
¼d.	= ¼	„	6	5	2¼
19s. 5¼d.			£5839	19	11¼

This Ex. differs from the preceding, only in having the price above 10s. In such a case we always take 10s. as the first aliquot part; and the remaining shillings and pence as aliquot parts, either of 10s., or of any amount which has been used in the course of the example. Sometimes it may happen that a second aliquot part of £1 is taken, as in 16s. 8d., where I should take 10s. = ½£, and 6s. 8d. = ⅓£.

Exs. 39.

			s.	d.
1.	18147	at	10	6½.
2.	3501	„	13	9.
3.	6234	„	11	4½.
4.	7646	„	16	0¾.
5.	5431	„	15	11¼.
6.	11040	„	16	10.
7.	2067	„	17	8¾.
8.	3459	„	18	6.

			s.	d.
9.	1948	at	19	0.
10.	8218	„	15	10½.
11.	12327	„	17	10¾.
12.	12497	„	19	10¼.
13.	14839	„	18	0¼.
14.	4103	„	14	10½.
15.	12018	„	19	5¼.
16.	8972	„	19	9¼.

Ex. IV. 7111 at £1 17s. 4½d.

Price £1 0s. 0d.		gives	7111
10s. 0d.	= ½£	„	3555 10
5s. 0d.	= ½	„	1777 15
2s. 0d.	= $\frac{1}{10}$£	„	711 2
4d.	= $\frac{1}{6}$	„	118 10 4
½d.	= $\frac{1}{8}$	„	14 16 3½
£1 17s. 4½d.			£13288 13 7½

Here £7111 is the value of 7111 things at £1 each; therefore, if I find the value of 7111 at 17s. 4½d., as in the last Ex., and add in the top line as £7111, I shall obtain a correct result.

Exs. 40.

		£	s.	d.				£	s.	d.
1.	1032 at	1	11	5½.	7.	11000 at		1	4	9¾.
2.	8649 „	1	0	6¼.	8.	1572 „		1	12	11½.
3.	15432 „	1	10	1.	9.	7538 „		1	11	8¼.
4.	6666 „	1	8	0½.	10.	19345 „		1	13	5.
5.	5741 „	1	17	6.	11.	1543 „		1	19	7¼.
6.	7891 „	1	14	11.	12.	8754 „		1	19	10¼.

Ex. V. 4013 at £12 7s. 0½d.

			4013
			12
A price £12 0s. 0d.		gives	48156
5s. 0d.	= ¼ £		1003 5
2s. 0d.	= $\frac{1}{10}$£		401 6
½d.	= $\frac{1}{48}$		8 7 2½
£12 7 0½			£49568 18 2½

Here, since £4013 repeated twelve times gives the value of 4013 at £12; therefore I must multiply the 4013 by 12, and add the product as pounds to the other amounts obtained by proceeding with the 7s. 0½d., as in Exs. III. and IV. The last division, by 48, cannot of course be performed mentally: a pupil may obtain the result by Long Division, and merely write down the amount.

Exs. 41.

		£	s.	d.			£	s.	d.
1.	6241 at	3	8	7¼.	10.	5432 „	17	15	10¾.
2.	999 „	4	19	9¼.	11.	7701 „	6	3	7¾.
3.	5683 „	5	17	10.	12.	7032 „	11	3	3¼.
4.	1429 „	8	19	10¼.	13.	7702 „	10	17	6½.
5.	4108 at	17	17	11½.	14.	3764 „	18	14	7½.
6.	10101 „	9	9	10.	15.	5505 „	9	19	11½.
7.	864 „	20	17	6.	16.	2807 „	29	0	7½.
8.	1875 „	13	13	7.	17.	11078 „	35	15	9.
9.	4273 „	18	17	8½.	18.	8970 „	63	14	8.

Ex. VI. 6583$\frac{8}{15}$ at £1 19s. 11$\frac{3}{4}$d.

A price £1	0	0		gives	6583		
	10	0	= $\frac{1}{2}$£		3291	10	
	5	0	= $\frac{1}{2}$		1645	15	
	4	0	= $\frac{1}{5}$£		1316	12	
		10	= $\frac{1}{6}$ of 5s.		274	5	10
		1	= $\frac{1}{10}$		27	8	7
		$\frac{1}{2}$	= $\frac{1}{2}$		13	14	3$\frac{1}{2}$
		$\frac{1}{4}$	= $\frac{1}{2}$		6	17	1$\frac{3}{4}$
$\frac{8}{15}$ of 1	19	11$\frac{3}{4}$	=........		1	1	3$\frac{13}{15}$
A price 1	19	11$\frac{3}{4}$		gives	£13160	4	2$\frac{7}{60}$

This Ex. differs from Exs. IV. and V. only in the presence of the fraction $\frac{8}{15}$. I therefore, after having proceeded with the sum, as though the $\frac{8}{15}$ were not there, find the value of $\frac{8}{15}$ at (£1 19s. 11$\frac{3}{4}$d.) *i.e.* of $\frac{8}{15}$ of (£1 19s. 11$\frac{3}{4}$d.) which, according to the method of Ex. IV. in (84) = £1 1s. 3$\frac{13}{15}$d.

The value of the $\frac{8}{15}$ might also have been thus obtained:

$$\frac{8}{15}=\frac{5}{15}+\frac{3}{15}=\frac{1}{3}+\frac{1}{5};$$

I might therefore have taken one-third and one-fifth of £1 19s. 11$\frac{3}{4}$d. and their sum would have amounted to £1 1s. 3$\frac{13}{15}$d., as before.

Exs. 42.

		£	s.	d.				£	s.	d.
1.	14287$\frac{1}{3}$ at	10	13	10$\frac{3}{4}$.		5.	4538$\frac{3}{8}$ at	0	16	6.
2.	3779$\frac{5}{16}$ „	14	4	4$\frac{3}{4}$.		6.	1008$\frac{7}{12}$ „	4	8	6.
3.	8976$\frac{3}{7}$ „	7	15	11$\frac{1}{2}$.		7.	2711$\frac{9}{16}$ „	7	3	3.
4.	4149$\frac{2}{11}$ „	0	8	7.		8.	3714$\frac{5}{14}$ „	2	9	11$\frac{1}{4}$.

Ex. VII. 4176 at £3 5s. 4$\frac{9}{10}$d.

					4176		
					3		
A price £3	0	0		gives	12528		
	5	0	= $\frac{1}{4}$£	„	1044		
		4	= $\frac{1}{15}$	„	69	12	
		$\frac{1}{2}$	= $\frac{1}{8}$	„	8	14	
		$\frac{1}{5}$	= $\frac{1}{20}$ of 4d.		3	9	7$\frac{1}{5}$
		$\frac{1}{5}$	= $\frac{1}{20}$ of 4d.		3	9	7$\frac{1}{5}$
£3	5	4$\frac{9}{10}$			£13657	5	2$\frac{2}{5}$

The presence of the fraction $\frac{9}{10}$d. is the only point in which this Ex. differs from III. and IV.; and just as we break up $\frac{3}{4}$d. into $\frac{1}{2}$d. and $\frac{1}{4}$d. so this $\frac{9}{10}$d. must be broken up into such portions as will be aliquot parts of 1d. viz. $(\frac{5}{10}+\frac{2}{10}+\frac{2}{10})$d., or $\frac{1}{2}$d. + $\frac{1}{5}$d. + $\frac{1}{5}$d.

If in this Ex. there had been a fraction at the end of the 4176, as in Ex. VI., we should have proceeded with it just as with the $\frac{8}{15}$ in that Ex.

Exs. 43.

		£	s.	d.				£	s.	d.
1.	5189 at	1	10	2$\frac{1}{7}$.		4.	2486$\frac{1}{3}$ at	0	18	7$\frac{1}{3}$.
2.	7485 „	4	5	9$\frac{3}{8}$.		5.	4321 „	1	0	6$\frac{1}{3}$.
3.	1111 „	14	6	5$\frac{3}{8}$.		6.	4231 „	11	8$\frac{1}{3}$	

Ex. VIII. 9lb. 3oz. 14dwts. at £10 15s. 6d. per lb.

Hitherto we have had the quantities whose value was required expressed all in one den^n; and we could therefore repeat the highest den^n of the price, as for instance, £1, as many times as there are units in the given quantity, and then take parts of this highest den^n for the remainder of the price. But in Ex. VIII. we cannot place 9lb. 3oz. 14dwts. in the top line, and multiply it by the 10, because the result would not be £10 repeated an exact number of times: I therefore place the £10 15s. 6d. in the top line, and multiplying it by 9, I obtain the value of 9lb. at £10 15s. 6d. per lb.; and the value of the 3oz. 14dwts. will be found, by taking the same parts of £10 15s. 6d. that 3oz. 14dwts. are of 1 lb. Thus working, we have

			£	s.	d.
			£10	15	6
					9
Value of 9lb. 0oz. 0dwt.		is	96	19	6
3 0	$= \frac{1}{4}$ of 1 lb.	„	2	13	$10\frac{1}{2}$
10	$= \frac{1}{6}$ of 3oz.	„		8	$11\frac{3}{4}$
2	$= \frac{1}{5}$ of 10dwt.	„		1	$9\frac{11}{20}$
2	$= \frac{1}{5}$ of 10dwt.	„		1	$9\frac{11}{20}$
9lb. 3oz. 14dwt.	=		£100	5	$11\frac{7}{20}$

$$\left(\frac{11}{20}+\frac{11}{20}+\frac{3}{4}+\frac{1}{2}\right)\text{d.} = \frac{11+11+15+10}{20}\text{d.}$$

$$= \frac{47}{20}\text{d.} = 2\tfrac{7}{20}\text{d.}$$

Also, since 3oz. 14dwt. $= 3\frac{14}{20}$oz. $= 3\frac{7}{10}$oz. $= \frac{3\frac{7}{10}}{12}$lb. $= \frac{37}{120}$lb., therefore the Ex. may be written $9\frac{37}{120}$lbs. at £10 15s. 6d. per lb.; and it is then similar to Ex. VI.

Exs. 44.

		£	s.	d.	
1.	5lbs. 10 oz. (Avoirdupois)....at	0	6	6	per lb.
2.	93 lbs. 3 oz. 1 dwt. 6 grs......„	4	10	6	„
3.	800 cwt. 0 qr. 16 lbs. ....„	14	19	$6\frac{1}{4}$	per cwt.
4.	99 tuns 3 hhds. 16 gals. ....„	128	18	9	per tun.
5.	155 gross and 75 ....„	1	10	6	per gross.

			£	s.	d.	
6.	135 quarters 7 bushels	at	2	17	8	per quarter.
7.	6 tons 7 cwt. 2 qrs. 17 lbs....	"	3	10	7	per cwt.
8.	8 years 3 months 20 days....	"	5	7	$6\frac{3}{8}$	per year.
9.	15 reams 9 quires 6 sheets ..	"	1	6	9	per ream.
10.	46 days 11 hours 35 minutes..	"	1	1	$5\frac{1}{2}$	per day of 12 hours.

Exs. 45.

MISCELLANEOUS EXAMPLES.

1. What cost 1351 lbs. at 2s. $2\frac{1}{4}$d. per lb.?
2. A bankrupt pays 13s. 9d. in the pound upon £1575, how much money did he divide?
3. Find the value of $2078\frac{1}{4}$ yards at 3s. $7\frac{3}{8}$d. per yard.
4. What is the tax on £12345 15s. at 3s. $7\frac{1}{2}$d. in the pound?
5. Find the worth of 24150 rupees at 1s. $11\frac{3}{4}$d. each.
6. A gold snuff-box weighed 7 oz. 15 dwts. 15 grs., find its value at £4 5s. 6d. per oz.
7. If 2s. 3d. in the pound is paid on an income of £1050; what is the net annual income?
8. My daily expenses are 10s. $11\frac{3}{4}$d.; how much can I save out of an income of £250?
9. Find the cost of a silver epergne weighing 175oz. 14dwts., at 45s. 9d. per oz.
10. What is the value of 3r. 17p. $25\frac{1}{4}$ yds., at £125 per acre?

131. In many Exs. similar in principle to those given above, a knowledge of fractions will enable a pupil to employ very brief methods of working; as, for instance, if I had 317 at 16s. 8d., I should say

$$317 \text{ at } 16\text{s. } 8\text{d.} = 317 \text{ at } \frac{5}{6}£ = \frac{1585}{6}£ = 264\tfrac{1}{6}£ = £264\ 3\text{s. } 4\text{d.}$$

Again, to find the value of 754 at 7s. 7d.

$$754 \text{ at } 7\text{s. } 6\text{d.} = 754 \text{ at } \frac{3}{8}£ = \frac{2262}{8}£ = 282\tfrac{3}{4}£;$$
$$= £282\ 15\text{s.}$$

and 754 at 1d. = 754d. = 62s. 10d.

therefore 754 at 7s. 7d. = £282 15s. + £3 2s. 10d.

= £285 17s. 10d.

This method is especially worth notice in short Exs.

Thus, 97 at $7\frac{1}{2}$d. $= 97 \times \frac{5}{8}$sh. $= \frac{485}{8} = 60\frac{5}{8}$sh. $=$ £3 0s. $7\frac{1}{2}$d.

Again, when the price is an even number of shillings, under 20, as 327 at 16s., we may work as follows:

327 at 16s. $= 327 \times \frac{8}{10}$£ $= \frac{2616}{10}$£ $= 261\frac{6}{10}$£ $=$ £261 12s.

and this mode of working is comprised in the following rule—

Multiply the given number by half the given price, doubling the first figure to the right-hand for shillings, and calling the rest pounds.

$$\text{Thus,}\quad \begin{array}{r} 327 \\ 8 \\ \hline \text{£}261\ 12\text{s.} \\ \hline\hline \end{array}$$

We may also observe, that since the cost of 12 things at 1d. each = 1s., therefore, that of 12 things at $3\frac{1}{2}$d. = $3\frac{1}{2}$sh. = 3s. 6d., and of 12 things at $7\frac{3}{4}$d. = $7\frac{3}{4}$sh. = 7s. 9d. *i. e.* if I have the price of one article in pence and a fractional part of a penny, the price of 12 articles will be expressed by the same figures as shillings and parts of a shilling.

Hence also the value of any multiple of 12 things may be readily expressed as above, if the price of one be given in terms of pence.

Ex. Find the value of 96lbs. at $10\frac{1}{4}$d.

Cost of 12lbs. at $10\frac{1}{4}$d. $= 10\frac{1}{4}$s. $=$ 10s. 3d.

therefore, cost of 8×12lbs. at $10\frac{1}{4}$d. $= 8 \times$ (10s. 3d.) $=$ 82s.

$=$ £4 2s.

With a little practice, such an Ex. as this might be worked mentally, more quickly than it could be written.

APPLICATIONS OF PROPORTION.

RULE OF THREE.

The Articles marked thus (*) may be omitted by those who have not read Fractions.

132.* In articles (65) to (76) the subject of Proportion has been fully discussed: and our business now is, to show how to work Exs., which are in reality only different forms of the question asked in (73), viz. If three numbers be given, what fourth number must be chosen, such that the four, when taken in order, shall be proportionals?

133. The Rule of Three is so called because in the questions given under this head there are three quantities proposed. These three numbers, when placed in order for working an Ex., are called *terms*. In using the expression first, second, and third *terms*, we intend to indicate the manner in which these three given quantities are arranged. Thus, if I were to write 9, 8, 24, as first, second, and third *terms* respectively, in the sense here intended, I should have

1st		2nd		3rd
9	:	8	::	24

where the dots placed between the terms are signs which express the relation existing among these terms, so that with them and the fourth term a Proportion is formed. When any three terms are properly arranged at the commencement of an Ex. in Rule of Three, they are said to

form a *Statement* : and the question is said to be stated. So long as only three terms are contained in a statement, the question is said to be one of *Simple Proportion* : but sometimes more than three quantities require to be so arranged: the question is then said to come under the head of *Compound Proportion*, or, as it is sometimes called, Double Rule of Three.

134.* Observing (73) and (74), we learn the following relations between the four terms forming any Proportion.

1st. That the third and fourth terms are of the same kind, *i. e.* that the third term must always be of the same nature as the one required.

2ndly. That the first and second must always be of the same kind; and that they must be reduced to the same denomination, if they be not already so expressed.

3rdly. That the fourth term is obtained by multiplying the third term by the fraction $\frac{\text{2nd}}{\text{1st}}$; or, if this operation be performed at two steps, we multiply by the second term, and divide by the first.

4thly. That if the second term be greater than the first, the fourth term will be greater than the third; but if the second term be less than the first, then the fourth term will be less than the third.

5thly. That since the fraction $\frac{\text{2nd}}{\text{1st}}$ is an abstract number, therefore the fourth term, which $= \frac{\text{2nd}}{\text{1st}} \times \text{3rd}$, is of the same denomination as that in which the third was expressed.

Collecting together these facts, we deduce the following Rule.

135. RULE. Find out the nature of the quantity sought by the question; *i. e.* of the required fourth term.

Of the three quantities given in the question, find that which is of the same nature as the fourth term, and take that quantity as the third term.

In order to place the other two quantities in their proper situations, inquire whether, from the nature of the question, the fourth term will be more or less than this third term: if *more*, make the *larger* of the two remaining quantities the middle or second term; but if *less*, make the *smaller* the middle term: the only remaining quantity must of course fill the first place.

If the first and second terms be not expressed in the same denomination, reduce them till they become so: and if the third term consist of several denominations, reduce it to the lowest name mentioned.

Then multiply the second and third terms together, and divide the product by the first: the quotient will be the answer or fourth term, expressed in the same denomination as that in which the third term was left.

If this quotient be expressed in too low a denomination, [as, for example, 1257 farthings, or 1836 dwts.,] let it be reduced to a higher denomination: [as £1 6s. $2\frac{1}{4}$d. and 7lb. 7oz. 16dwts.]

136. I will now proceed to work some Exs. which will illustrate the different varieties that may be expected under the head of Simple Proportion: and it will be found that the principal difficulty consists in arranging the three terms according to the directions prescribed by the Rule. This is especially the case, when the question is given in such a shape, that the three terms cannot be *immediately* obtained from the question as it stands. I will explain this more

fully as I go through the various kinds of Exs. When the *Statement* is once obtained, the remainder of the work consists merely of Multiplication, Division, and Reduction.

I recommend a pupil to work every question that I have worked, so that he may the better see the correctness and ascertain the object of the successive operations in any Example.

Ex. I. If 12 yards of cloth cost £19, what will 8 yards cost?

Here I see that the required fourth term will be money; I therefore place the £19 in the third term. Also, the fourth term, which is to be the price of 8 yards, will be *less* than the third term, which is the price of 12 yards; therefore, according to the Rule, I place the *smaller* of the two remaining terms, *i. e.* the 8 yards, in the middle, and the 12 yards in the first place.

```
yds.   yds.    £
12  :  8  ::  19
               8
         12) 152
           £12 13s. 4d.
```

The first and second terms are already in the same name, and the third term contains but one denomination, therefore no reduction is required. I now multiply the second and third terms together, and divide by the first: the answer is £12 13s. 4d. And this fourth term and the other three terms form the following proportion;—

12 yds. : 8 yds. :: £19 : £12 13s. 4d.

or in words, 12 yards are to 8 yards, as £19 are to £12 13s. 4d.

Ex. II. If 17cwt. 3qrs. and 14lbs. cost £8 18s. 9d. how much may be bought for £5 12s. 6d. at the same rate?

The fourth term will evidently be expressed in weight; therefore I put 17cwt. 3qrs. 14lbs. in the third term. I now ask this question: "If the quantity in the third term can be obtained for £8 18s. 9d., will more or less be bought for the £5 12s. 6d.?" evidently *less*; therefore I place the *less* of the two prices in the middle, and the remaining one first. The first and second terms are not expressed in any single denomination; I therefore reduce them to threepences, which is the highest denomination to which they can both be reduced. Also, since the third term consists of more denominations than one, I reduce it to the lowest denomination mentioned, viz. lbs. After I have multiplied the second and third terms together, and divided by the first, the quotient is 1260, which consists of lbs., because the third term was expressed in lbs. This quotient, when reduced to higher denominations, becomes 11cwt. 1qr.

```
 £  s. d.       £  s. d.       cwts. qrs. lbs.
 8 18  9    :   5 12  6   ::    17    3    14
   20             20             4
  ---            ---            --
  178            112            71
    4              4            28
  ---            ---           ---
  715            450           582
  ===            ===          142
                              ----
                              2002
                               450
                             ------
                             100100
                             8008
                      715) 900900 (1260lbs.
                           715
                           ----
                           1859
                           1430            (4) 1260
                           ----         28 {   -----
                            4290           ( 7) 315
                            4290                ---
                            ----              4)45
                               0              -----------
                            ====              11cwt. 1qr.
                                              ===========
```

I will now give a few Exs. in which the difficulty consists in preparing the question for being stated; but I shall merely show how to overcome the difficulty, and leave the question to be worked out as in the former Exs.

Ex. III. A bankrupt's effects amounted to £980 10s., and he paid his creditors 13s. 4d. in the pound; what was the amount of his debts?

At first sight there appear to be only two terms in this question, but the £1 furnishes another term. Now all the three quantities are money; but by reading the question thus: "If 13s. 4d. be paid for a debt of £1, what debt will be paid by £980 10s.?" I learn that 13s. 4d. and £980 10s. are money *paid*, and the £1 is money *owed*, or *debt*; and since the fourth term is *debt*, I place the £1 in the third term. Also, the debts which are paid by £980 10s. are of course more than this third term; therefore I place the larger term, £980 10s., in the middle, and 13s. 4d. in the first place. The statement will then be 13s. 4d. : £980 10s :: £1. The fourth term will be found to be £1470 15s.

137.* The four terms, arranged as a proportion, will be

13s. 4d. : £980 10s. :: £1 : £1470 15s.

or, as two equal ratios—

$$\frac{13\text{s. }4\text{d.}}{£980\ 10\text{s.}} = \frac{£1}{£1470\ 15\text{s.}}$$

which, expressed in words, indicates that the payment, 13s. 4d., is the same portion of the whole payment £980 10s. that the debt of £1 is of the whole debt £1470 15s.

138. **Ex. IV.** How much may a person spend in 73 days, if he wishes to lay by every year 50 guineas out of an income of £450?

Since I wish to know how much may be spent in 73 days, I must ascertain how much is spent in one year: I therefore subtract the 50 guineas, which are saved, from the whole income of £450: the remainder is 397 10s., and the question now becomes—"If in 365 days I spend £397 10s., how much may I spend in 73 days?" The statement will be

$$\begin{array}{ccccc} & & & \pounds & s. \\ & & & 450 & 0 \\ \text{days.} & & \text{days.} & \underline{52} & \underline{10} \\ 365 & : & 73 & :: & 397\ \ 10 \end{array}$$

and the required sum will be found to be £79 10s.

Ex. V. If the sixpenny loaf weigh 3lbs. when wheat is 6s. per bushel, what should it weigh when wheat is 6s. 9d. per bushel?

This seems a very simple question, but most pupils make an error in the statement; for, after correctly placing the 3lbs. in the third term,—when they ask the question, as to whether the answer will be more or less than this term, they reason that if the loaf weigh 3lbs. when wheat is at 6s., it will weigh more when wheat is at 6s. 9d.; whereas, since the wheat is dearer, we ought to have a less weight of bread for the same money: and the statement will be

6s. 9d. : 6s. :: 3lbs.

Where questions are met with involving the necessity of finding the area of a surface, or the volume of a solid, we must refer to Art. Duodecimals.

139.* I will now by a few Exs. illustrate the use of Frac[ns].

Taking the statement already made in Ex. III., I have

$$\text{Fourth term} = \frac{\text{2nd} \times \text{3rd}}{\text{1st}} = \frac{(\pounds 980\ 10s.) \times \pounds 1}{13s.\ 4d.}$$

$$\left(\begin{array}{l}\text{when first and second are re-} \\ \text{duced to a fractional part} \\ \text{of a } \pounds.\end{array}\right) = \frac{980\tfrac{1}{2}\pounds \times 1\pounds}{\tfrac{2}{3}\pounds}$$

$$= \left(\frac{1961}{2} \times 1 \times \frac{3}{2}\right)\pounds$$

$$= \frac{5883}{4}\pounds = 1470\tfrac{3}{4}\pounds$$

$$= \pounds 1470\ 15s.$$

Again, in Ex. V.

$$\text{Fourth term} = \frac{6\text{s.} \times 3\text{lbs.}}{6\frac{3}{4}\text{s.}} = \frac{6 \times 3}{\frac{27}{4}}\text{lbs.} = \frac{\overset{2}{\cancel{6}} \times \cancel{3} \times 4}{\underset{\underset{3}{\cancel{9}}}{\cancel{27}}} = \frac{8}{3}\text{lbs.} = 2\tfrac{2}{3}\text{lbs.}$$

Ex. VI. If $4\frac{5}{9}$oz. avoirdupois cost $\frac{31}{32}$s., what will $8\frac{13}{24}$lbs. cost?

$$4\tfrac{5}{9}\text{oz.} : 8\tfrac{13}{24}\text{lbs.} :: 8\tfrac{31}{32}\text{s.}$$

$$\text{Fourth term} = \frac{8\frac{13}{24}\text{lbs.} \times 8\frac{31}{32}\text{s.}}{4\frac{5}{9}\text{oz.}} = \frac{\frac{205}{24} \times 16\text{oz.} \times \frac{287}{32}\text{s.}}{\frac{41}{9}\text{oz.}}$$

$$= \left(\frac{\overset{5}{\cancel{205}}}{\underset{8}{\cancel{24}}} \times \frac{\cancel{16}}{1} \times \frac{287}{\underset{2}{\cancel{32}}} \times \frac{\overset{3}{\cancel{9}}}{\cancel{41}}\right)\text{sh.} = \frac{15 \times 287}{16}\text{s.}$$

$$= \frac{4305}{16}\text{s.} = 269\tfrac{1}{16}\text{s.}$$

$$= £13\ 9\text{s.}\ 0\tfrac{3}{4}\text{d.}$$

Questions wherein fractional quantities occur in the terms may also be worked by the use of Decimals; but this is not an advantageous method of solving them, especially when some of the decimals are circulating. See Ex. VI. in (119).

Exs. 46.

1. If 24 men earn £36, what sum will 42 men earn at the same rate?

2. In how many days will 5 guineas be spent, at the rate of 7 shillings in 3 days?

3. A wall containing 372 square feet is paid for at the rate of 1s. $10\frac{1}{2}$d. per square yard; find the cost of the whole.

4. How many yards of paper, 27 inches wide, will hang a room 54 feet round and 10 feet high?

5. If an acre is 220 yards in length, and 22 in breadth, what must be the length when the breadth is $27\frac{1}{2}$ feet?

6. If I can buy $15\frac{3}{4}$ yds. of cloth for 10 guineas, how much can I buy at the same rate for £283 17s. 6d.?

7. An income of £150 pays a tax of £4 7s. 6d.; what will be the tax upon £586 1s.?

8. If the carriage of $15\frac{1}{4}$ cwt. for 56 miles come to 10s. 6d., how much can I have carried 72 miles for the same money?

9. The carriage of 4cwt. for 72 miles cost 15s. 9d., how many lbs. can be carried $13\frac{3}{4}$ miles for the same sum?

10. A field of 16 acres produces 440 bushels of wheat: how much is that upon every 22 square yards?

11. A creditor agreeing to receive £51 for a debt, finds that he has been paid at the rate of 12s. 9d. in the pound; how much was the debt?

12. Three men who weave at the rate of 5¼ yds. per day, finish 119 yds. in a certain time: at what rate per day must they weave, who finish 85 yds. in the same time?

13. If 15 men eat 35 shillings worth of bread in a certain time, when wheat is 12s. per bushel; how much may they eat at the same cost when wheat is 7s. per bushel?

14. A rate of 2s. 9d. in the pound produces £352; what is the rental of the parish?

15. If an income tax of 10d. in the pound amounts to 2¼ millions: what must be the poundage in order to produce £3,375,000?

16. A man buys 148 yds. at 2s. 7½d. per yd.; at what price must he sell it to gain £12 12s. 10d. by the whole?

17. The penny loaf weighs 10 oz. when wheat is 5s. a bushel; how much will it weigh when wheat is 6s. 3d.?

18. The rental of a parish is £5626 10s.; and the assessment for the poor-rates is £405 13s. 4d.; how much will be the rate on £45 12s. 9d.?

19. If the net income of an estate after paying all taxes be £534 15s., and the gross income be £570 8s.; how much in the pound did the taxes amount to?

20. The chain for measuring land is 4 perches in length, and is divided into 100 links; what is the length of a wall in feet, which measures 1550 links?

21. If 1 lb. of gold, and 1oz. of alloy, can be coined into 44½ guineas; find the value of 5oz. of pure gold, considering the alloy of no value.

22. I spend 12 guineas in 35 days, and save £100 a year: what must I earn in the year?

23. A piece of gold at £3 17s. 10½d. per oz. is worth £150: what will be the worth of a piece of silver of equal weight, at 54s. 6d. per lb.?

24. Two floors are equal in size, one is 35 feet long, and 25 feet broad; the other is 40 feet long; what is its breadth?

25. The weights of gold and of water are as 19¼ and 1; find what number of solid inches of gold is equal in weight to 17½ cub. ft. of water?

26. A dollar is to a crown as 111 : 120; how many dollars are equal in value to £250?

27. A clock which gains 7½ minutes in 24 hours, is 14 minutes fast at Monday midnight; what time will it indicate at 6 o'clock in the evening of the following Thursday?

28. A person owes £1537 3s. 4d.; but can pay only £960 14s. 7d.: what will be the dividend, and how much shall I receive for a debt of £276 11s. 6d.?

29. The shadow of a stick 3 ft. 6 in. long is 2 ft. 9 in.: what is the height of a tree which at the same time throws a shadow of 154 feet.—(See App. Art. Proportion.)

30. An income of £3827 12s. 6d. is taxed at the rate of 7d. in the pound; how much clear income will remain?

31. Bought 236 gallons of oil for £111 6s. 8d.; what profit will be made by selling it at the rate of 8s. 6d. for 3 quarts?

32. Paid £45 10s. for a hogshead of rum; how much water must be added, to be able to sell it without loss or gain at 11s. 6d. per gallon?

33. Out of an income of £312 10s. a year, the expenses are £55 in 146 days; in what time will 1000 guineas be saved?

34. If 90 English degrees correspond to 100 French degrees, how many French degrees are there in 36·45 English?

35. What must be the breadth of a piece of ground which is 14$\frac{3}{8}$ yds. long, so that it may be as large as a piece 40$\frac{1}{8}$ yds. long and 4$\frac{3}{4}$ broad?

36. The 6d. loaf weighs 3$\frac{1}{4}$ lbs., when wheat is 50s. a quarter; what will it weigh, when wheat is 40s. 3d. a quarter?

37. Given that the velocity of a falling body is proportional to the time during which it falls; find the time of descent of a body having acquired a velocity of 1000 feet, supposing that the velocity obtained in 2$\frac{1}{2}$ seconds is 80·5 feet.

38. The lengths of the arms of a lever are inversely proportional to the weights at the extremities of the arms; if the lengths of the arms be 3 feet and 2$\frac{1}{4}$ inches, what must be the weight at the longer arm to balance 20 lbs. at the shorter end?

39. A bankrupt payed £1520 to his creditors; £205 of his debts were paid in full; and his assets were to his debts as 3 : 8; find the amount that he owed.

40. What can a man save per annum, who out of an income of £500 gives away $\frac{1}{12}$th, pays 7d. in the pound income tax, and spends £16$\frac{3}{8}$ in 3 weeks?

COMPOUND PROPORTION.

140. It was observed in (133) that a question was classed under the head of Compound Proportion, when there were more than three quantities which required to appear in the statement. The following is an Example.

Ex. I. If 12 yards of cloth, 3 quarters wide, cost £19, what will be the cost of 8 yards, 5 quarters wide?

If the width of the two pieces of cloth were the same, we should take no account of this width, whatever it might be: and the question would then

become Ex. I. in (133); or one of Simple Proportion. Referring to that Ex. we have the statement

$$12 \text{ yards} \;:\; 8 \text{ yards} \;::\; £19 \qquad \text{(B)}$$

and the cost of the new piece $= \frac{8 \text{ yds.} \times £19}{12 \text{ yds.}} = \frac{8 \times 19}{12} £.$

We will now take into account the two breadths, 3 qrs., and five qrs., and put the following question—"If a piece of cloth cost $\frac{8 \times 19}{12}£$, when 3 quarters wide, what will it cost when the width is 5 quarters?" The statement would be

$$3 \text{ qrs.} \;:\; 5 \text{ qrs.} \;::\; \frac{8 \times 19}{12} £ \qquad \text{(C)}$$

and the fourth term $= \frac{8 \times 19}{12} £ \times \frac{5 \text{ qrs.}}{3 \text{ qrs.}} = \frac{8 \times 19 \times 5}{12 \times 3} £.$

Now, if the whole of statement (B), and the first and second terms in (C), be converted into one statement, as follows—

$$\begin{matrix} 12 \text{ yds.} \\ 3 \text{ qrs.} \end{matrix} \;:\; \begin{matrix} 8 \text{ yds.} \\ 5 \text{ qrs.} \end{matrix} \;::\; £19 \qquad \text{(D)}$$

and we take the product of the two quantities which stand first, as our first term, and the product of the two in the middle, as our second term we shall have

$$\text{the fourth term} = \frac{8 \text{ yds.} \times 5 \text{ qrs.} \times 19£}{12 \text{ yds.} \times 3 \text{ qrs.}} = \frac{8 \times 5 \times 19}{12 \times 3} £,$$

which is precisely the same as was obtained from the two successive statements: hence such a statement as (D) will produce a correct result. Also, in forming this statement, independently of (B) and (C), I select for the third term that which is similar to the required fourth term, as in Simple Proportion: and in placing the remaining terms, I take each pair separately, and ask the usual question with the third term, as to whether the answer will be more or less than this term; and I arrange this pair precisely as though they were the only two terms which I had to consider. However many pairs of terms occur in the question, they must all be treated in like manner; for the same proof that has shown how to combine the first and second statements, will show how to combine the third with the result of the first pair.

I will work another Ex. and mention the mental operations which must be performed, in order to enable me to place each pair of terms correctly.

Ex. 11. A field, 300 yards long and 280 broad, was ploughed by six horses in two days of eight hours each; how many horses will plough a piece of ground 500 yards long, and 315 broad, in three days of ten hours each?

The fourth term will be horses: I therefore place the six horses in the third term. Also, the new field is longer than the one which required six horses; hence, considering the effect of this pair of terms alone, it will require more horses, and I place the larger term, 500, in the middle: so, also, the second field is broader than the first, and therefore will take more horses, and I place 315 in the middle. Again, the first field was ploughed in two days, but the new one in three days; hence, since the time is longer, we shall, so far as this pair of terms is concerned, require fewer horses; and the smaller term, two, is to be in the middle. Also, in ploughing the first field, the days were eight hours, but for the second field they are ten hours; hence, with this extra time, fewer horses will be required, and I place the eight in the second place. The whole statement is as annexed; and

$$\left.\begin{array}{lcl} 300 \text{ yards} & & 500 \text{ yards} \\ 280 \text{ yards} & : & 315 \text{ yards} \\ 3 \text{ days} & & 2 \text{ days} \\ 10 \text{ hours} & & 8 \text{ hours} \end{array}\right\} :: 6 \text{ horses}$$

$$\text{the fourth term} = \frac{\overset{5}{\cancel{500}} \times \overset{\cancel{9}\,3}{\cancel{315}} \times 2 \times \cancel{8} \times \overset{2}{\cancel{6}}}{\underset{\cancel{3}}{\cancel{300}} \times \underset{\cancel{35}}{\cancel{280}} \times \cancel{3} \times \underset{\cancel{2}}{\cancel{10}}} \text{ horses}$$

$$= 3 \times 2 \text{ horses} = 6 \text{ horses};$$

precisely the same number as before: *i. e.* the increased size of the field, and the increased length of time allowed for the work, are so balanced, that the same number of horses as before is sufficient.

In arranging the several pairs of terms in the statement, I seem to be trying at one time to obtain a smaller term, and at another time a larger term than the third; and it is true that some conditions of the question tend to make the fourth term less than this third, and some to make it more. Each pair will produce its own effect in increasing or diminishing the required term; and we shall therefore find the result more or less than the third, according as the conditions in the question which would make it more, predominate, or not, over those which would make it less;

i. e. according as the product of all the terms in the second place is more or less than the product of all in the first place.

It must be carefully observed, that if any pair of corresponding terms be not expressed in the same denomination, they must be so reduced, just as in Simple Proportion, before we commence forming the fraction which will give the fourth term.

Exs. 47.

1. If 10 men can dig 30 yds. of earth in 8 days, how many yards can be dug by 20 men in 4 days?

2. If £300 gain £10 in a year, in what time will £900 gain £175 10s.?

3. Three boats take 6000 herrings in 8 days; in how many days will 450 boats take 20,000 barrels, each containing 700 herrings?

4. I borrow £175 10s. for 10 months, when money is worth 5 per cent.; how much must I lend in return for 12 months, when money is worth $3\frac{1}{2}$ per cent.?

5. If five men can reap a field whose length is 800 feet and breadth 700, in $3\frac{1}{2}$ days of 14 hours each; in how many days of 12 hours each can seven men reap a field whose length is 1800 feet, and breadth 960 feet?

6. The papering of a room $10\frac{1}{3}$ feet high, and 20 yards round, cost £1 2s. 6d.; what will be the cost of papering another room 9 feet high, and 63 feet round?

7. If I pay 1s. 3d. for 6lb. 14oz. of bread, when wheat is 4s. 9d. per bushel, what must I pay for 23lb. 12oz., when wheat is 5s. 5d. per bushel?

8. A printing machine turns out 37,260 sheets in a day, running $12\frac{1}{2}$ hours; if its speed be increased in the ratio of 4 to 3, how many sheets will be wrought in $7\frac{1}{2}$ hours?

9. A carriage wheel, the circumference of which is $16\frac{1}{2}$ feet, and which makes 45 revolutions per minute, goes 275 miles in a certain time; how many revolutions per minute must a wheel make, to perform 385 miles in the same time, the circumference of the latter wheel being $19\frac{1}{4}$ feet?

10. A field of 12 acres having 120 stalks to each square yard, and 70 grains to each stalk, produces wheat to the value of £96 16s.: what will be the worth of the produce of 800 square yards, having 175 stalks to the square yard, and 45 grains to each stalk?

11. An iron beam 16 ft. long, $2\frac{1}{4}$ ft. broad, and 8 in. thick, weighs 1280 lbs.: what must be the length of a beam whose breadth is $3\frac{1}{4}$ ft., thickness $7\frac{1}{2}$ in., and weight 2028 lbs.?

12. If a wheel which revolves at the rate of 470 times in 8 minutes,

ts interest, for any length of time, is called the
ıt.

ən interest is paid only upon the sum originally lent, alled *Simple* Interest; but when at the end of any greed upon, as for instance a year, the interest is to the principal, so that this amount forms the al for the next year; and a similar addition is made end of every such period, then it is termed *Compound* st.

The questions which occur in this Rule are merely les of Proportion.

. Find the Simple Interest of £382 10s. for one year at 5 per n other words—If the principal £100 give £5 interest, what vill be derived from the principal £382 10s.? the statement will r be

£100 : £382 10s. :: £5.

, it will be found that every Ex. in Simple Interest nish a similar statement, in which we observe that t term is £100, the second is the principal, and the s the rate.

now multiply the second term by the 5, and divide 100; *i. e.* we multiply by the rate and divide by 100. these Exs. we do not, as usual in the Rule of Three, the first and second terms to the lowest denomina- pressed in either of them, but multiply by 5 and by 100, as in Compound Multiplication and Division. m, when worked in the usual form, stands thus:

```
£100  :  £382 10s.  ::  £5
                 5
    1,00) 19,12 10
               20
             2,50
               12
             6,00  Answer.  £19 2s. 6d.
```

make 50 revolutions in a certain time; how many revolutions will another wheel make in the same period, at the rate of 360 revolutions in 7 minutes?

13. If 15 men eat 13s. worth of bread in 7 days, when wheat is 12s. per bushel; what should be the price so that 10 men should be furnished for $12\frac{1}{2}$ days at the same cost?

14. A hay-field which has $2\frac{1}{2}$ tons to the acre is mown by 20 men in 6 days working 8 hours a day; what number of hours per day must 13 men work for 8 days upon a field which has $3\frac{1}{4}$ tons to an acre?

15. The circumferences of the smaller and larger wheels of a carriage are in the ratio of 5 to 6. Let the carriage move in a ring, so that the circumferences of the circles described by the inner and outer wheels shall be as 7 : 8. Given that the inner large wheel makes 800 revolutions in describing $\frac{1}{7}$ of its path, find the number of revolutions made by the small outer wheel, while describing $\frac{1}{8}$ of its path.

16. If the price of 100 bricks, of which the length, breadth, and thickness are 16, 8, and 10 respectively, be 5s. 4d.; what will be the price of 9760 bricks, which are one-fourth greater in every dimension?

17. If 5 steam engines of 9-horse power (when employed 3 days a week, and 10 hours a day) raise through a certain altitude 25 three-bushel sacks of wheat, weighing 60 lbs. a bushel; in what time will 9 engines of 8-horse power (when employed 5 days in the week, and 9 hours a day) raise through 15 times the former altitude, 75 two-bushel sacks of wheat, weighing 63 lbs. a bushel?

18. Three fire engines, each having 4 pipes, 3 square inches in section, are worked at the rate of 20 strokes in 3 minutes, and discharge 4680 gallons of water in 16 minutes; how many engines, each having 3 pipes, 5 square inches in section, and worked at the rate of 17 strokes in $2\frac{1}{2}$ minutes, will discharge 20000 gallons in half an hour?

INTEREST.

141. INTEREST is the payment made for the use of money for any time, and is generally reckoned at so many pounds a-year for £100 lent; or, as it is commonly called, so many pounds per cent. For instance, if £5 be the interest of £100 lent for a year, we should say that the money is lent at the rate of 5 per cent. per annum.

The sum lent is called the *Principal;* the interest of £100 for one year the *Rate;* and the sum lent, together

with its interest, for any length of time, is called the *Amount*.

When interest is paid only upon the sum originally lent, it is called *Simple* Interest; but when at the end of any time agreed upon, as for instance a year, the interest is added to the principal, so that this amount forms the principal for the next year; and a similar addition is made at the end of every such period, then it is termed *Compound* Interest.

142. The questions which occur in this Rule are merely Examples of Proportion.

Ex. I. Find the Simple Interest of £382 10s. for one year at 5 per cent.; in other words—If the principal £100 give £5 interest, what interest will be derived from the principal £382 10s.? the statement will evidently be

£100 : £382 10s. :: £5.

Also, it will be found that every Ex. in Simple Interest will furnish a similar statement, in which we observe that the first term is £100, the second is the principal, and the third is the rate.

We now multiply the second term by the 5, and divide by the 100; *i. e.* we multiply by the rate and divide by 100. And in these Exs. we do not, as usual in the Rule of Three, reduce the first and second terms to the lowest denomination expressed in either of them, but multiply by 5 and divide by 100, as in Compound Multiplication and Division. The sum, when worked in the usual form, stands thus:

£100 : £382 10s. :: £5
5
1,00) 19,12 10
20
2,50
12
6,00 *Answer.* £19 2s. 6d.

Here the division by 100 is performed by cutting off two ciphers at the end of the divisor, and two figures at the end of the dividend: and the remainder after each division is reduced, as in Compound Long Division. I have written the 100 as a divisor, but in practice it is omitted.

143. Since in (141) we multiplied by 5, and divided by 100, therefore we might at once have multiplied by the fraction $\frac{5}{100}$, or $\frac{1}{20}$: that is, the operation of finding the interest might have been performed mentally by dividing by 20: and this is generally done when the rate is 5 per cent., but not otherwise.

144. If the interest for any number of years is required, multiply the interest for one year by the number of years. If for any number of months or weeks, aliquot parts of the interest for one year may be taken, as in Practice: but if for any number of days, it should be found by Proportion.

Ex. II. Find the interest of £175 for three years and 135 days, at five per cent.

The interest of £175 for one year is £8 15s., and for three years is 3 × (£8 15s.) = £26 5s. Now, to find what is the proportionate amount of interest for 135 days, we have this statement;

days.		days.		£	s.
365	:	135	::	8	15

and the answer is £3 4s. $8\frac{52}{73}$d.: therefore the whole interest = £26 5s. + £3 4s. $8\frac{52}{73}$d. = £29 9s. $8\frac{52}{73}$d.

Exs. of this kind, involving Simple Interest for years and days, may also be worked as follows:

Since we obtain Simple Interest for 1 year by multiplying by the rate per cent., and dividing by 100,

$$\therefore \text{ in this case, S. Int}^{t} \text{ for 1 year} = 175 \times \frac{5}{100}\,£ \qquad \text{(E)}$$

and expressing the 135 days as a fractional part of a year, and multiplying the interest of 1 year by the number of years, viz. $3\frac{135}{365}$, or $3\frac{27}{73}$

$$\therefore \text{ whole interest} = 175 \times \frac{5}{100} \times 3\frac{27}{73} = 175 \times \frac{5}{100} \times \frac{246}{73}$$

$$= \frac{215250}{7300} = £29\ 9s.\ 8\tfrac{52}{73}d.$$

Obs. I did not reduce the fr^n $\frac{5}{100}$ in (E) to lower terms, because the den^r is generally more simple, when the 100 is left uncancelled.

Under the head of Simple Interest may be included all questions generally classed under the heads of Commission, Brokerage, and Insurance; for all such quantities are calculated at a fixed rate for every £100.

Exs. 48.

Find the Interest of

	£	s.	d.		yr.		£	
1.	324	0	0	for	1	at	5	p. c.
2.	475	10	6	„	1	„	4	„
3.	875	12	3	„	1	„	3¼	„

Find the Amount of

	£	s	d.		yrs.		£	
4.	1025	0	0	for	3	at	2¾	p. c.
5.	1750	9	0	„	1½	„	3½	„
6.	1827	18	9	„	6½	„	2¼	„

Interest of

	£	s.	d.		y.	m.		£	
7.	540	17	6	for	1	5	at	4	p.c.
8.	1845 guineas			„	3	10	„	3	„

Amount of

	£	s.	d.		y.	m.		£	
9.	237	10	0	for	4	11	at	5¼	p.c.
10.	11428	0	0	„	9	7	„	1¾	„

Interest of

	£	s.	d.		y.	w.		£	
11.	755	6	8	for	1	15	at	2¾	p.c.
12.	935	13	4	„	3	39	„	4½	„

Amount of

	£	s.	d.		y.	w.		£	
13.	1875	5	0	for	3	45	at	4¼	p.c.*
14.	2000	0	0	„	11	30	„	3½	„

Interest of

	£	s.	d.		y.	d.		£	
15.	1440	15	0	for	1	73	at	5	p. c.
16.	2500	0	0	„	3	90	„	4	„

Amount of

	£	s	d.		y.	d.		£	
17.	1175	2	6	for	4	300	at	3¼	p.c.
18.	990	11	0	„	7	150	„	6	„

145. As an Ex. in Compound Interest we may take the following question:—

Ex. III. What is the amount of £350 in three years at five per cent. Compound Interest?

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* In this and the following Exs. the fractional parts of a penny have been neglected when finding the interest for the weeks.

Putting down only the results of the three operations, we have

	£	s.	d.
Principal of first year	350	0	0
Interest of first year	17	10	0
Amount of first year, and principal of second year	367	10	0
Interest of second year	18	7	6
Amount of second year, and principal of third year	385	17	6
Interest of third year	19	5	$10\frac{7}{8}$
Amount at the end of third year	405	3	$4\frac{3}{8}$

Exs. 49. Find, at Compound Interest,

The Interest of

	£	s.	d.	yrs.	£	
1.	1500	0	0 for	4	at 5	p. c.
2.	354	13	6 „	$2\frac{1}{2}$	„ $4\frac{1}{2}$	„
3.	1820	15	0 „	$3\frac{1}{4}$	„ $2\frac{1}{2}$	„

The Amount of

	£	s.	d.		£	
4.	750	0	0 for	6 *y.* at....	$5\frac{1}{2}$	p. c.
5.	2025	0	0 „	3 *y.* 150 *d.* at	5	„
6.	1825	11	6 „	4 *y.* 7 *mo.* „	3	„

146. Questions may be found in Interest which involve some little difficulty, because there do not appear at once three terms out of which to form a statement. And most pupils will find that in any difficult question involving the application of Proportion, as in Profit and Loss, Stocks, &c., they cannot succeed in thoroughly comprehending it, without placing it in a plain Rule of Three form. Take for instance

Ex. IV. In what time will £75 12s. 6d. amount to £99 16s. 6d. at four per cent. per annum?

Here the interest gained is found by subtracting the principal, £75 12s. 6d., from the amount, £99 16s. 6d., and it = £24 4s. Also, the interest of £75 12s. 6d. for one year at 4 per cent. is £3 0s. 6d. Hence I have this question:

If £75 12s. 6d. produce £3 0s. 6d. in one year, in what time will it produce £24 4s.? The statement is

£3 0s. 6d. : £24 4s. :: 1 year,

and the fourth term will be found to be 8 years.

Ex. V. What sum will amount to £104 2s. 6d. in four years at $4\frac{1}{2}$ per cent. simple interest?

I must here inquire what £100 would amount to in four years at $4\frac{1}{2}$ per cent.; the answer is £118. The question is now, therefore,

If £100 become £118, what sum will in the same time amount to £104 2s. 6d.? The statement is

£118 : £104 2s. 6d. :: £100

and the required fourth term is £88 4s. $9\frac{57}{59}$d.

Ex. VI. At what rate per cent. will £152 10s. amount to £191 7s. 9d. in six years?

I have in this Ex. to find the interest of £100 for one year. Now the interest gained by £152 10s. in six years is (£191 7s. 9d. – £152 10s.), or £38 17s. 9d., or in one year £6 9s. $7\frac{1}{2}$d.: hence the question now is,

If £152 10s. gain £6 9s. $7\frac{1}{2}$d., what will £100 obtain? The statement will be

£152 10s. : £100 :: £6 9s. $7\frac{1}{2}$d.

and the fourth term will be found to be £4 5s., *i.e.* the rate per cent. is $4\frac{1}{4}$.

Exs. 50.

1. In what time will £62 amount to £71 6s. at 5 per cent. per annum?*

2. In what time will £1215 15s. amount to £1291 14s. $8\frac{1}{4}$d. at the rate of $2\frac{1}{2}$ per cent.?

3. For how many years must I put out £987 12s. to interest at $4\frac{1}{4}$ per cent., in order that I may receive £1197 9s. $3\frac{3}{8}$d.?

4. What sum of money will in 1 year amount to £108 13s. 6d. at $3\frac{1}{2}$ per cent.?

5. Required the principal which will in 3 yrs. amount to £136 2s. $1\frac{1}{8}$d., at $2\frac{1}{4}$ per cent.

6. Find what sum will produce interest amounting to £330 15s. in 7 yrs. at $4\frac{1}{2}$ per cent.

7. At what rate of interest will £95 15s. amount to £112 10s. $1\frac{1}{2}$d. in 5 yrs?

8. What must be the per centage in order that £1175 may become £1637 13s. $1\frac{1}{2}$d. in $7\frac{1}{2}$ yrs.?

9. Required the rate per cent at which 1000 guineas will gain £590 12s. 6d. in $12\frac{1}{2}$ yrs.

10. Find the rate per cent. at which any sum of money will double itself in 8 yrs.

11. What sum lent at 5 per cent. Compound Interest will in 3 years amount to £358 17s. $3\frac{3}{10}$d.?

* Simp le Interest is always implied, unless the contrary be expressed.

12. Find the difference between the Simple and Compound Interest of £416 13s. 4d. for 2 yrs. at 2½ per cent.

13. What is the commission on £20500 at 3¼ per cent.?

14. Find the premium on a policy of life insurance for £2500 at £5 17s. 9d. per cent.?

15. What annual premium must a farmer pay on stock valued at £370 10s., at 2s. 3d. per cent.?

16. In what time will £818 18s. 4d. amount to £1064 11s. 10d. at 3¾ per cent.?

17. At what rate of interest will £732 10s. amount to £1245 5s. in 10 yrs.?

18. If the interest on £130 15s. 10d. for 10 days be 3s. 7d., how much is that per cent. per annum?

147. Questions concerning Annuities, Leases, and Reversions involve applications of Interest; but they generally require for their solution either algebraical expressions or tables derived from these. The following is however a simple example of the kind.

Ex. VI. What is the amount of an annuity of £50 left unpaid for 5 yrs., allowing Compound Interest at 4 per cent. per annum?

I write down merely the outlines of the work, and neglect all sums below 1d., as the fractions obtained after two or three divisions become exceedingly heavy.

	£	s.	d.
Amount due at the end of first year	50	0	0
Interest due at the end of second year	2	0	0
Add £50, due at the end of second year	50	0	0
Principal of third year	102	0	0
Interest at the end of third year	4	1	7
Add £50 due at the end of third year	50	0	0
Principal of fourth year	156	1	7
Interest at the end of fourth year	6	4	10
Add £50, due at the end of fourth year	50	0	0
Principal of fifth year	212	6	5
Interest at the end of fifth year	8	9	10
Add £50, due at the end of fifth year	50	0	0
Amount due at the end of five years	270	16	3

If Simple Interest alone were allowed, I should write the interest for the successive years by itself, and add its amount to the final amount;—by this means no interest would be allowed upon interest, *i. e.* there would be *no Compound* Interest.

The subjoined Examples are worth notice.

Ex. VII. What must I give for a freehold, let for £225 a year, so as to have 4½ per cent. for my money? Or in other words,

If every £100 laid out bring £4½, what sum will produce £225? The statement will be,

$$£4\tfrac{1}{2} \;:\; £225 \;::\; £100$$

and the fourth term $= \dfrac{225 \times 100}{4\frac{1}{2}}£ = \overset{25}{\cancel{225}} \times 100 \times \dfrac{2}{\cancel{9}}£ = £5000$

148. Sometimes in speaking of the price of a piece of property, it is said that a certain number of years' purchase is given for it: this is the same as so many years' rental. Thus, if a field, the rent of which is £4, be sold for £100, we say that 25 years' purchase was given for it, because the price is 25 times the rental.

Ex. VIII. How many years' purchase should be paid for freehold property to clear 4½ per cent.?

I must here see how many times a rent of £4½ must be repeated to produce £100, the price of the land which gives £4½.

This number $= \dfrac{£100}{£4\frac{1}{2}} = 100 \times \dfrac{2}{9} = \dfrac{200}{9} = 22\tfrac{2}{9}$; the price paid is therefore said to be $22\tfrac{2}{9}$ years' purchase.

Exs. 51.

1. What will an annuity of £60, payable yearly, amount to in 6 yrs. at 5 per cent. Compound Interest?
2. Find the amount due from a pension of £100, payable half-yearly, which has been unpaid for 3½ yrs., allowing 5 per cent. Comp. Interest?
3. What principal lent for 2½ yrs. at 5 per cent. Compound Interest will amount to £700 12s. $9\frac{3}{10}$d.?
4. What should be the purchase money of an estate, of which the rental is £5200, so that the buyer may receive 3½ per cent. for his money?
5. A purchaser invests £7500 in land, and receives 2¾ per cent. upon his investment; what is the rent?
6. What per centage is received upon the purchase money, when an estate whereof the rent is £367 10s., is bought for £10500?
7. How many years' purchase should be paid for freehold property, to produce 3¾ per cent.?

8. A freehold is sold at 33 years' purchase; what rate of interest is received on the investment?

9. What is the value of a perpetual annuity of £120 at the rate of $4\frac{1}{4}$ per cent.?

10. Property which brings 7 per cent. lets for £85 15s.; what was the purchase money?

DISCOUNT.

149. DISCOUNT is an allowance made by a creditor to a debtor who pays a debt before it is due. When this allowance is subtracted from the debt, the remainder, *i. e.* the sum that is paid, is called the present worth.

Discount is calculated at a certain rate per cent, and in common usage is treated just the same as Interest: we shall, however, show that this is not strictly correct, but that the person who pays the money has thereby more than the just allowance made to him.

For instance; if £50 were due to me at the end of one year, but I were willing to allow a discount of 5 per cent. for ready money, then according to the common usage, I should throw off the interest of £50 for one year, viz. £2 10s., and receive only £47 10s. But if I make a creditor an allowance for paying ready money, I do so upon the supposition that I can place out to interest the ready money which I receive, and together with the interest can make up the £50 at the end of the year. Now, if I put out to interest £47 10s. at 5 per cent., I shall obtain as interest £2 7s. 6d., and therefore I shall in all receive £49 17s. 6d.: hence I lose 2s. 6d. by this arrangement. The real question now is—What sum put out to interest for a year at 5 per cent. will amount to £50? Or—If £100 will

amount to £105, what sum will amount to £50? The statement will be

$$£105 \;:\; £50 \;::\; £100 \qquad (F)$$

and the fourth term will be found to be £47 12s. 4$\frac{4}{7}$d,

Also, the interest of £47 12s. 4$\frac{4}{7}$d, for one year is £2 7s. 7$\frac{3}{7}$d.; and this, together with the principal, = £50 : and therefore I neither gain nor lose.

By observing (F) we notice that the third term, £100, is the present worth of the first term, £105; and the fourth term is the present worth of the second term, £50: and in any question where the present worth of a sum is required, the third term is £100; the first term is the amount of £100 at interest for the given time; and the middle term is the sum due.

150. If the discount, and not the present worth, be required, we must place in the third term the discount of £105, viz. £5. But since the discount in the third term would generally require to be reduced to the lowest denomination expressed, and the work be thereby rendered heavy; it is therefore generally better to find the present worth, and then obtain the discount by subtracting the present worth from the bill due.

151. The most common form in which discount occurs is in the use of what are called *Bills*, which are stamped papers, bearing a written engagement to pay a sum of money at a certain future time. If such a bill be presented to a banker before it is due, *i. e.* before the time fixed for payment, and the persons who are responsible for this bill are considered able to meet it at the proper time, the banker will give ready money for it, retaining,

however, the *discount* upon the sum, as his remuneration for the accommodation.

In practice, as was said, it is usual to charge interest, and not discount: therefore the banker gains by the transaction, and the amount of this gain will be found to be the interest upon the true discount. For if we refer to the Ex. in (149) we shall see that in discounting a bill of £50 due in twelve months, the banker would deduct £2 10s., *i.e.* the interest on £50; whereas he ought to have deducted only £2 7s. $7\frac{3}{7}$d., which is the interest upon £47 12s. $4\frac{4}{7}$d.; therefore the extra sum which he takes is the interest upon £2 7s. $7\frac{3}{7}$d., or the interest upon the true discount.

Though discount is not in practice correctly used, yet a pupil in working Exs. should always employ the true method.

The bills mentioned above are said to be *drawn* upon the person or persons who agree to pay the money, and those who allow any such bill to be drawn on them are said to *accept* it: hence they are called acceptors, and the bill itself is called an acceptance. These acceptances are generally for any number of calendar months: but in this country three days, call *Days of Grace,* are allowed after the bill is nominally due, before it is legally due; so that a bill drawn on March 30th, at three months, would not be legally due till July 3rd.

Ex. II. What does a banker gain by discounting a bill of £403 4s., drawn Oct. 13, at four months, and discounted, Dec. 5, at 4 per cent?

Here the bill is legally due on Feb. 16, and from Dec. 5 to Feb. 16 are 73 days:

		£	s.	d.
The interest for that time ..	=	3	4	$6\frac{18}{125}$
And the true discount	=	3	4	0
Therefore, the banker's gain	=			$6\frac{18}{125}$d.

Exs. 52.

Find the Discount on

	£	s.	d.	months	£
1.	100	0	0	for 6	at 5 per ct.
2.	128	18	6	„ 3	„ 4 „
3.	157	10	0	„ 9	„ 6 „
4.	1128	17	6	„ 7	„ $3\frac{1}{2}$ „

Find the present worth of

	£	s	d.	months	£
5.	1000	0	0	due in 12	at 5 per ct.
6.	875	10	0	„ 8	„ $4\frac{1}{2}$ „
7.	119	6	0	„ 5	„ $3\frac{1}{8}$ „
8.	425	15	6	„ $2\frac{1}{2}$ y.	at 3 „

9. Find the difference between the interest and discount of £525 for $1\frac{1}{2}$ yrs. at 5 per cent.

What would a banker gain by discounting the following bills?—

		Drawn	Discounted	
10.	£325 8s. 4d.	March 15 at 4 months,	April 6th,	at 4 per cent.
11.	£90 7s. 6d.	Sept^r 1 „ 9 „	Jan. 15th,	„ 5 „

12. What is the present worth of £500, one half of which is due in 4 months, and the remainder in 6 months, discount at $5\frac{1}{8}$ p. c. per ann.?

152. A very important application of Discount occurs in those branches of business, where it is the custom to take off 20, 30, &c., per cent. discount from the gross or invoice price of goods. And great errors may sometimes be made by tradesmen who not know how much to add to the net value or an article, in order that they may, without loss, make the deduction agreed upon.

For instance, suppose a tradesman has an article of which the net price should be 50s., and it is usual to allow 20 per cent. discount, or deduct one-fifth from the invoice price. If he, thinking to allow for this discount, puts on $\frac{1}{5}$th of the 50s., and thus makes a gross price of 60s.; then, when he takes off 20 per cent. or $\frac{1}{5}$th, he will find his net price to be 48s., thereby losing 2s. But if, instead of putting on $\frac{1}{5}$th he had put on $\frac{1}{4}$th, he could then take off the required $\frac{1}{5}$th, and be no loser.

			s.	d.
Thus,	Net price required	=	50	0
	Add $\frac{1}{4}$th of 50s.	=	12	6
	Gross price	=	62	6
	Subtract $\frac{1}{5}$th of 62s. 6d. ..	=	12	6
	Net price, as before	=	50	6

The general rule will be found as follows: Let the discount agreed upon be represented as a fractional part of £100: Thus, for 5, 10, 20, 30, 35, &c. per cent., the fractions would be

$$\frac{5}{100}, \frac{10}{100}, \frac{20}{100}, \frac{30}{100}, \frac{35}{100}, \text{ \&c.}$$

or, in their lowest terms,

$$\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{7}{20}, \text{ \&c.} \qquad \text{(G)}$$

These are the fractional parts of the gross price to be *deducted* from it: and the corresponding frac[l] parts that must be first *added* to the net price will be

$$\frac{1}{20-1}, \frac{1}{10-1}, \frac{1}{5-1}, \frac{3}{10-3}, \frac{7}{20-7}, \text{ \&c.}$$

$$\text{or} \quad \frac{1}{19}, \frac{1}{9}, \frac{1}{4}, \frac{3}{7}, \frac{7}{13}, \qquad \text{(H)}$$

where it is to be observed that the fractions in (H) have numerators the same as in (G); but the den[rs] are the former den[rs], minus the respective numerators.*

* The following demonstration of the above rule may be read by those acquainted with the elements of Algebra.

Let G = gross price; N = net price; also let x represent the frac[l] part of N to be added thereto to produce G, so that $N + xN = G$; also let $\frac{m}{n}$ represent the frac[l] part of G to be taken off as discount so as to leave N: so that $G - \frac{m}{n}G = N$; it is required to find x.

$$N + xN = G, \text{ or } \quad N(1+x) = G, \quad \text{ or } N = \frac{G}{1+x} \qquad (1)$$

$$\text{also } G - \frac{m}{n}G = N \quad \text{ or } N = \left(1 - \frac{m}{n}\right)G \qquad (2)$$

Equating these two values of N, we have

$$\frac{G}{1+x} = G\left(1 - \frac{m}{n}\right) \quad \text{ or } \frac{1}{1+x} = 1 - \frac{m}{n} = \frac{n-m}{n}$$

$$\text{inverting both sides, } 1 + x = \frac{n}{n-m}$$

$$\text{and } x = \frac{n}{n-m} - 1 = \frac{n-n+m}{n-m} = \frac{m}{n-m}$$

i.e. if we wish to take off as discount $\frac{m}{n}$, we must put on $\frac{m}{n-m}$; and these two fractions represent the series of fractions in (G) and (H).

For example, if 30 per cent. discount is to be allowed upon an article of which the net price should be 21s., we refer to the rows (G) and (H), and learn that if 30 per cent. or $\frac{3}{10}$ is to be taken off, $\frac{3}{7}$ must be put on: and we have as follows.

		s.
Net price....................	=	21
Add $\frac{3}{7}$ths of 21s.............	=	9
Gross price charged in invoice...	=	30
Subtract $\frac{3}{10}$ths of 30s........	=	9
Net price, as before, ...	=	21

PROFIT AND LOSS.

153. PROFIT AND LOSS is another application of Proportion. It is calculated at so much per cent., or at a certain *per centage*, and the general object of all Exs. under this head is, to find—(1) What per centage of profit or loss will result from selling an article at a certain price:—(2) At what price must it be sold, that there may arise a certain per centage of profit and loss; the prime cost of the article being in both cases known.

154. It will not be attempted to exhibit an Ex. of every kind of question that may arise; but a sufficient number will be given to show the principles upon which all the questions depend; and the particular method of applying the principles of Proportion in each case must be left to the judgment of the pupil.

Ex. I. If an article cost £2 7s. 3d., and be sold for £3 3s. 0d., what is the gain per cent.?—or, If £2 7s. 3d. become £3 3s. 0d., what will £100 become? The statement is

£2 7s. 3d. : £100 :: £3 3s.

and the fourth term is £133 6s. 8d. Hence the gain upon £100 is £33 6s. 8d.; or the profit is at the rate of 33⅓ per cent.

Ex. II. If, by selling tea at 6s. 4d. per lb., a grocer lose 6 per cent., what was the prime cost per lb.?

Now, to lose 6 per cent. is to obtain only £94 for every £100 laid out; hence the question is really this—If £100 be laid out, and £94 be received for is, what is laid out when 6s. 4d. is received?

Here, since £100 is prime cost, or buying price, and we want buying price in the fourth term, we have this statement:

£94 : 6s. 4d. :: £100

and the fourth term, or prime cost of 1lb. is 6s. 8$\frac{40}{47}$d.

Ex. III. At what price must I sell a commodity purchased at the rate of £14 5s. per cwt. so as to gain 21 per cent.?

In this Ex. it is required to receive £121 for £100 laid out: therefore £121 is the selling price of that which cost £100; and since the fourth term is to be the selling price of 1cwt., we have

£100 : £14 5s. :: £121

and the required price is £17 4s. 10$\frac{1}{5}$d.

Questions of this kind, wherein we require the price at a certain profit per cent., may often be worked more briefly by the method of Practice. Thus to gain 20 per cent. on any sum of money invested is merely to add one-fifth of the sum to the previous amount; and Ex. III. may be worked as follows.

	£	s.	d.
	14	5	0
Add 20 per cent. or $\frac{1}{5}$th	2	17	0
Add 1 „ or $\frac{1}{20}$th of 20 p. c..		2	10$\frac{1}{5}$
∴ original sum + 21 per cent. profit =	£17	4	10$\frac{1}{5}$

The following Ex. involves the principles of Exs. II. and III.

Ex. IV. A person, by disposing of goods for £182, loses at the rate of 9 per cent.; what should have been the selling price, so as to make a profit of 7 per cent.

We may work this question by two operations: first, find the prime cost, and then from it find that selling price which would give a profit of 7 per cent. Since to sell at 9 per cent loss is to receive but £91 for that which cost £100, we have this statement:

£91 : £182 :: £100

and the answer is the prime cost £200. Also, to obtain a profit of 7 per cent. is to receive £107 for that which cost £100; therefore, to find the selling price of that which cost £200, we have

£100 : £200 :: £107

and the answer is £214, the price at which the goods should be sold to make 7 per cent profit.

In order to work the question by one statement, we may put it under this form—If goods sold for £182 bring £91 for every £100 laid out, what ought they to be sold for, so as to bring £107 for every £100?

Here we have the £100 laid out the same in both circumstances, and it will therefore not affect the question: £107 and £91 may be considered as the gaining and losing rates; also, £182 is the sum received for goods, and therefore of the same nature with the fourth term: hence the statement is

£91 : £107 :: £182;

and the fourth term, or selling price, is £214, the same that was obtained from the former statements.

Exs. 53.

1. Paid £137 12s. 6d. for goods, and sold them for £151 7s. 9d.; what was the profit per cent.?

2. By selling goods at 3s. 6d. I gain 12 per cent.; what shall I gain or lose by selling them an 4s. 9d.?

3. I give 3s. 9d. for goods; at what rate must they be sold to make a profit of 30 per cent.?

4. Bought cloth at 9s. 4½d. per English ell; it is required to find the selling price per yard so as to gain 17½ per cent.

5. Goods were bought for 2s. 9d., being 17½ per cent. below their real value; what was that value?

6. Sold goods for £3 13s. 6d., being 22½ per cent. profit; what was the prime cost?

7. I sell an article for £22 10s., and by so doing lose 15 per cent.; what per centage would be lost or gained by selling it at £27?

8. At a selling price of 15s. I lose 10 per cent.; what must be the price to gain 10 per cent.?

9. I buy tobacco at 10 guineas per cwt.; at what price must I retail it per lb. so as to gain 12 per cent.?

10. When the price of a certain article is 12s. 6d. there is a gain of 25 per cent.; what would be the loss or gain if the price were 10s.?

11. I bought 145 quarters of wheat at 50s. per quarter, and in selling I make a profit of £36 5s.; how much per cent. was the profit?

12. A merchant sold a pipe of wine for £50, and by so doing lost 5 per cent.; at what price must he sell 3 other pipes so that he may gain 5 per cent. upon the prime cost of the 4 pipes?

13. A person having bought goods for £20, sells half of them so as to gain 10 per cent.; for how much must he sell the remainder so as to gain 20 per cent. upon the whole?

14. I bought 56 gallons of brandy at 22s. 6d. per gallon, but 7 gallons were lost; at what price per gallon must I sell the remainder, to obtain 15 per cent. profit on the whole outlay?

PARTNERSHIP.

155. Partnership, or as it is sometimes called Fellowship, is the Rule by which we determine how to divide profits, which arise from different sums of money put into a business by two or more persons, either for the same or different periods of time.

Ex. I. Two persons enter into business as partners; one puts in £350, and the other £500; they gain £100. How is the profit to be divided?

Here the profit, £100, is made from the whole capital, £850; and each partner's share of the profit will be in proportion to his share of the capital: therefore the question divides itself into these two parts:—(1) If £850 produce a profit of £100, how much will £350 produce? (2) If £850 produce £100, what will £500 produce?

The statements for these two questions will plainly be

	£		£		£
(1)	850	:	350	::	100
(2)	850	:	500	::	100

		£	s.	d.
and the fourth term of (1)	=	41	3	$6\frac{6}{17}$
" " " (2)	=	58	16	$5\frac{11}{17}$
and these together......	=	100	0	0

Ex. II. A field of grass is rented by two persons for £27; the one keeps in it 15 oxen for ten days, and the other 21 oxen for seven days: find the rent to be paid by each, supposing the pasturage to remain equally good throughout?

Here it is plain that the keep of 15 oxen for ten days is the same as of ten times 15, or 150, oxen for one day: so also, of 21 oxen for seven days is the same as of 7 times 21, or 147, for one day; therefore the question is plainly this: If one man turn into a field 150 oxen, and another 147, for one day, and they together pay £27, how is that payment to be divided?

The whole number turned in would be 297, and the two statements would be similar to those in Ex. I., viz.

$$297 : 150 :: £27 \quad (1)$$
$$297 : 147 :: £27 \quad (2)$$

the fourth term of (1) $= \dfrac{150 \times \cancel{27}}{\underset{11}{\cancel{297}}}£ = £13\ \ 12s.\ \ 8\tfrac{8}{11}d.$

„ „ „ (2) $= \dfrac{147 \times \cancel{27}}{\underset{11}{\cancel{297}}} = £13\ \ 7s.\ \ 3\tfrac{3}{11}d.$

and their sum.................. = £27 0 0

156. I will give one more Ex. which is the same in principle as Exs. I. and II., but is more complicated in its operations.

Ex. III. On the 1st of January *A* brought into a business £350, and on the 1st of April £500 more: on the 1st of June he takes out £400; three months after this he brought in £600. *B* brought into the business £500: four months after this he takes out £150; and on the 1st of November he brought in £650. At the end of the year their clear gain is £1008. How much ought each to receive?

Here *A* put in £350 from January 1st to June 1st, or five months; also, £500 from April 1st to June 1st, or two months: he has now in the business £850, but he takes out £400, leaving £450. This £450 is in from June 1st to December 31st, or seven months. Also, he has £600 in from September 1st to December 31st, or four months. Hence he has in all

£			£		
350 for	5	months, or	1750	for 1	month
500 „	2	„ or	1000	„ 1	„
450 „	7	„ or	3150	„ 1	„
600 „	4	„ or	2400	„ 1	„
Therefore he has in all			8300	„ 1	„

Again, B brought in £500 from January 1st to May 1st, or four months: he now takes away £150, and has in £350 from May 1st to December 31st, or eight months. Also, he brings in £650 from November 1st to December 31st, or two months. Hence he has

£		£
500 for 4 months,	or	2000 for 1 month.
350 „ 8 „	or	2800 „ 1 „
650 „ 2 „	or	1300 „ 1 „
Therefore B's capital is....		6100 „ 1 „
and A's capital was		8300 „ 1 „
therefore the joint capital =		14400 „ 1 „

Hence the two statements will be

	£		£		£
For A's share	14400	:	8300	::	1008
For B's „	14400	:	6100	::	1008

and the fourth terms are £581 for A, and £427 for B.

157. In this place we may introduce an Ex. of the following kind.

Ex. IV. A wine merchant mixes together 20 gallons of wine at 12s. a gallon, 25 gallons at 14s., and 36 gallons at 16s.: what should be the price of a gallon of the mixture?

Here it is plain that	20	gallons at	12s.	are worth	240s.
also „	25	„	14s.	„	350s.
and „	36	„	16s.	„	576s.
and therefore that	81	of the mixture		„	1166s.

hence the value of one gallon is plainly $\frac{1}{81}$th of the value of the whole;

or, price per gallon = $\frac{1166}{81}$s. = 14s. $4\frac{20}{27}$d.

158. The following Ex. shows how to divide a given quantity into parts which shall have to each other given ratios. It is upon the same principle as the previous Exs., though not commonly recognized as such.

Ex. V. Divide 1065 into parts which shall be to each other in the ratio of 3, 5, 7; and also into parts which shall be in the ratio of $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$.

Taking the former part of the Ex., we observe that $3 + 5 + 7 = 15$ is the smallest integer which can be divided in the ratio of 3, 5, 7; hence the

required portions of 1065 will bear the same ratio to 1065 that 3, 5, 7 do respectively to 15. The first portion is obtained by the statement

$$15 \ : \ 3 \ :: \ 1065;$$

and the fourth term = 213. Also, if 5 and 7 be successively placed in the second term of the above statement, we shall find the remaining portions to be 355, and 497.

Similarly, in the latter part of the Ex. we see that $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$, or $\frac{35+21+15}{105}=\frac{71}{105}$ is a fraction which can be readily broken up into fractions which shall be in the ratio of $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$; hence the required portions of 1065 will bear the same ratio to the entire number 1065 that $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{7}$, respectively bear to their sum $\frac{71}{105}$; we have, therefore as the statement for obtaining the first portion

$$\frac{75}{105} \ : \ \frac{1}{3} \ :: \ 1065;$$

and the fourth term $=\frac{\overset{\overset{5}{\cancel{15}}}{\cancel{1065}}}{1}\times\frac{1}{\cancel{3}}\times\frac{105}{\cancel{71}}=525$. So also the second and third statements will be

$$\frac{71}{105} \ : \ \frac{1}{5} \ :: \ 1065;$$

$$\text{and } \frac{71}{105} \ : \ \frac{1}{7} \ :: \ 1065;$$

and the corresponding fourth terms will be found to be 315 and 225.

Exs. 54.

1. Two persons invest in business £300 and £250 respectively; they gain £150: how is it to be divided?

2. *A* and *B* as partners lost £600; if *A's* capital were £4000, and *B's* £2100, how much of the loss must each bear?

3. *A*, *B*, and *C* were partners: *A* put in £1000 for 2 yrs., *B* £750 for 15 months, and *C* £1500 for 9 months; divide equitably a profit of £1000.

4. *A*, *B*, and *C* rent a field for £10; *A* put in 20 horses, *B* 15 oxen, and *C* 10 sheep; how should the expense be divided, if the eating of a horse, ox, and sheep be in the ratio of 3, 2, and 1?

5. *A* puts into a concern £500, and 6 months after puts in £300 more; *B* puts in £1000, and 3 months after puts in £1000 more; they trade for 2 yrs., and gain £650: what is the share of each?

6. Divide £20 amongst 3 persons, so that their shares shall be in the ratio of 3, 4, and 5.

7. Distribute £705 in the ratio of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.

8. A mixture is made of 5lbs. of tea at 3s. 4d., 10lbs. at 4s. 2d., and 15lbs. at 5s.: what should be the price per lb. of the mixture?

9. A wine merchant mixes together 20 gallons at 12s., 30 gallons at 15s., 40 gallons at 20s., and 10 gallons of water; what must be the retail price so as to gain 10 per cent. profit?

10. A testator bequeaths £1000 to one person, £500 to a second, and £300 to a third; but his property is found to realise only £1250: how much should each receive?

Exs. 55. I.

1. If 375 quarters are grown on 75 acres of land, how much land will be required to grow a peck?

2. Ten persons joined to buy 3 lottery tickets for £10, £25, and £40: the second gains a prize of £1000, how much does each man gain?

3. How many times does a clock tick in the month of January, if it ticks 15 times in 2 minutes?

4. A large bin contains 15 cub. ft. 267 cub. in., and out of it 3 smaller bins are filled, each containing 4 cub. ft. and 375 cub. in.; how much will be left in the bin?

5. Out of a square plate of metal $15\frac{3}{4}$ feet long, how many circular pieces can be cut $2\frac{1}{4}$ inches in diameter?

6. What is the smallest sum that a man must have in his pocket, that he may be able to pay it away entirely either in moidores, guineas, marks, or 7 shilling pieces?

7. Assuming the method of multiplying and dividing fractions by whole numbers, shew that $\frac{3}{5} \div \frac{5}{9} = \frac{27}{25}$.

8. If 9 slabs, 12 in. long, and 12 in. broad, will cover a certain surface, how many slabs will be needed if they be 18 in. long, and 8 in. broad?

9. Find the cost of 8725 lbs. at 1s. $10\frac{3}{4}$d.

10. What cost 25 oz. 3 dwts. 11 grs. at £3 17s. $10\frac{1}{2}$d. per oz.?

11. If a tradesman gain 1s. 6d. on an article which he sells for 5s. 9d., what is the profit per cent. on the prime cost?

12. Of two pieces of cloth, one is 42 in. wide, and costs 1s. 6d. per yd.; the other is 56 in. wide, and costs 2s. 4d. per yd.; what is the ratio of the qualities of the pieces, supposing the prices to be exactly proportionate to their real value?

13. Silver coinage has 37 parts pure silver, and 3 parts of copper; 1 lb. Troy weight makes 66s.; what quantity of pure silver is there in 20s.?

14. Reduce 8 cwt. $3\frac{1}{2}$ qrs. $1\frac{1}{2}$ stones to the decimal of 15 tons.

15. What decimal of a square furlong is 1 perch?

16. If the net price of an article be found by taking off $\frac{2}{15}$ths of the gross price, what must be added to the net price to make the invoice price?

17. If I am required to throw off $\frac{1}{15}$th, $\frac{2}{31}$ths, $\frac{3}{17}$ths, from the invoice price of three parcels of goods, what must be the fractional parts to be added to the required net prices?

K.

1. How many strokes will a clock strike in the month of May?
2. Find the value of 4424 articles at 15s. 10½d. each.
3. A block of stone 7 ft. long, 2½ ft. broad, and 15 in. thick, weighs 3¼ tons; what must be the length of a block of the same kind, whereof the breadth is 3¼ ft., the thickness 10 inches, and weight 6500 lbs.?
4. What is the cost of 59 tons, 15 cwt., 3 qrs., 18 lbs., at £26 18s. 9d. per ton?
5. A person buys 68 yds. of cloth for £75, and retails it at £1 18s. per English ell; what does he gain by the transaction?
6. What may a person spend per day out of an income of £1000 a year, if he lay by 20 guineas every calendar month?
7. Explain the reason of stating a Rule of Three sum; and shew why the answer results in the same name as the third term.
8. A tax of 3d. in the pound on a certain assessment produces £1080, how much will be produced by a tax of 7d. on an assessment of double the value?
9. What is the value of $\left(\frac{3}{5} + \frac{2}{3} - \frac{5}{12}\right)$ of £360?
10. Explain the nature and advantages of Decimal Fractions, compared with Vulgar Fractions.
11. What fraction of a square mile are 2¼ perches?
12. I have to distribute 150 yds. to 10 men and 10 women, so that the men and women may have shares in the ratio of 2 : 3; how much will each have?
13. A boat is propelled by 8 oars, which take 10 strokes per minute; and it goes 9 miles an hour; find the rate of a boat propelled by 6 oars which take 8 strokes per minute, when 5 of its strokes are equal to 6 of the former.
14. Find the exact value of

$$\cdot375 \text{ of } 6s.\ 8d. - \cdot941875 \text{ of } 4s. + 1\cdot9898\dot{3} \text{ of } 2s.$$

15. Simplify the following expression:—

$$\frac{1}{2} + \frac{1}{3 \times 2^3} + \frac{1}{5 \times 2^5} + \frac{1}{7 \times 2^7}$$

16. If I throw off 17 per cent. from my invoice price, what fractional part of my net price must I put on?

L.

1. The quotient is 3276, the divisor £2 7s. 6d., remainder 6d.; find the dividend.

2. On the 1st of March I borrow £10, to be repaid in a calendar month, and in return lend £15 on the 1st of April; when should it be repaid?

3. The girth of a tree at the surface of the ground is 6 feet, find the girths at 10 feet and 20 feet, if the height of the tree be 50 feet, and it tapers regularly.

4. Find the cost of $13\frac{7}{16}$ oz. at £3 7s. 6d. per oz.

5. At what distance from the end of a slab of 17 in. breadth must I cut, so as to have a rectangular piece, containing half a square yard?

6. An author pays 2s. 6d. for the printing, &c. of a book: out of the publishing price 10 per cent. is allowed for advertising, 10 per cent for publisher's commission, 25 p. c. to the retail trade, 4 p. c. for damaged copies, and 5 p. c. loss of interest; what must be the publishing price, so that he may neither gain nor lose?

7. In the last question, what would he gain on 2000 copies, at a selling price of 5s. 6d.?

8. If 56 current shillings be worth £$2\frac{7}{16}$ in gold, how many current shillings are worth £65 in gold?

9. Two numbers are to one another as 8 : 11; and the greater one is 77; find the less.

10. Find the whole cost of a house, of which the rent is £27; the poor-rate 3s. 4d. in the pound; gas rate two-thirds of the poor-rate; and the paving rate three-fifths of the gas rate.

11. Reduce to a vulgar fraction

$$\frac{2\cdot3}{1\cdot\dot{7}} \times \frac{13\cdot\dot{8}\dot{5}}{1\cdot0\dot{2}} \times \frac{1\cdot21}{4\cdot9}.$$

12. Express the ratio of £$3\cdot\dot{7}$ to $4\cdot1\dot{5}$ guineas in the smallest integers.

13. A piece of work employs 15 men for 6 days, when the day is 12 hours long, and costs £$18\frac{5}{8}$; what will be the cost of a piece employing 25 men for 9 days of 10 hours each, the pay of the new workmen being $1\frac{1}{3}$ times that of the old?

14. A tank is filled by 3 pipes in 2, $8\frac{3}{7}$, and $7\frac{1}{3}$ hours respectively; in what time would they all fill it?

15. A discount of $\frac{2}{19}$ths is agreed upon: what must be the ratio between the net and gross prices, so that I may be able to make the deduction?

16. I add $\frac{7}{13}$ths to the net price of an article; what per centage of discount was agreed upon?

STOCKS.

159. The government of a country sometimes finds it necessary to borrow money; and it gives to the lender a bond acknowledging the debt, and agreeing to pay a certain rate of interest for the money. The amount owing to those who hold these bonds is called the National Debt, or *the Funds;* and the interest paid is derived from the income of the country, arising principally from taxation.

The above-mentioned bonds are saleable, and are called *Stock;* and they of course vary in value, principally according to the plentifulness or scarcity of money.

Thus, suppose a person lend £100 to government, and receive an acknowledgment for it, with an agreement to pay £3 a year interest for the loan; then, if at any time he wishes to sell the bond, and money is scarcer than when he lent the £100, he will get less for it than he gave, perhaps £95; and if money be more plentiful, he will get more for it, perhaps £105. But still the bond represents an acknowledgment for £100, and £3 interest is paid to the holder of it, whatever may be the sum which he has paid for it. When, therefore, we say that 3 per cent. stock is selling at 80, we mean that the buyer of £100 bond, or as it is called, £100 stock, has to give only £80 sterling for it; hence, as he gets £3 interest for it, it is not £3 *per cent.* to him, but £3 for £80. From this it is plain that the lower the stock is in price, the better interest the buyer obtains; and the higher the stock is, the less interest he obtains for his money.

When the market price of £100 stock is exactly £100, the stock is said to be at *par;* if the price be more than £100, it is said to be at a *premium;* if below £100, at a

discount. The smallest variation in the price of stock is one-eighth of £1, or 2s. 6d., for every £100 stock.

160. Since persons who wish to sell stock may not know any who wish to buy, therefore all sales and purchases are transacted through agents, who are called Stock-brokers. The broker of the buyer deals with the broker of the seller, and each charges his employer, or principal, as he is called, a commission of $\frac{1}{8}$th of £1 for the transfer of every £100 stock; so that a buyer must always consider that he pays $\frac{1}{8}$th per cent. more, and the seller that he receives $\frac{1}{8}$th per cent. less than the selling price; but if in any Ex. the commission be not mentioned, no notice need be taken of it.

161. In working the various questions that occur in Stocks, a pupil must be careful not to confound stock and actual money. Also, in buying or selling stock, it is quite immaterial whether it be 3 per cent., 4 per cent., or any other kind of stock, unless we wish to know the *income* to be derived: for instance, if I have to sell out £100 stock, when the price is 95, it matters not whether it be in the 3, 4, or 5 per cents.: I have to receive £95 for every £100.

Ex. I. What must be given for £5050 stock in the Three per Cents., at $85\frac{3}{8}$ per cent.?

Here the price of £100 stock is £$85\frac{3}{8}$; and I have to find the price of £5050 stock: hence the statement will be

£100 : £5050 :: £$85\frac{3}{8}$.

Ex. IV. Multiplying by $85\frac{3}{8}$, after the method of (84, Ex. IV.), and then dividing by 100, we have the annexed operation, in which we find the cost of £5050 stock at $85\frac{3}{8}$ = £4311 8s. 9d.

		5050	
		$85\frac{3}{8}$	
		25250	
		40400	
$\frac{2}{8}$, or $\frac{1}{4}$, of £5050 =		1262	10
$\frac{1}{8}$ of £5050 =		631	5
	1,00)	4311,43	15
		20	
		8,75	
		12	
		9,00	

Next, let us find what amount of stock any given sum will buy, when the price of £100 stock is known.

Ex. II. How much stock can be bought for £1490, the price being 88$\frac{3}{8}$, and commission $\frac{1}{8}$ per cent.?

Adding the commission to the market price, we have the cost to the buyer 88$\frac{4}{8}$, or 88$\frac{1}{2}$. The term sought is amount of stock; and in the proposed question £100 is the stock to be bought by £88$\frac{1}{2}$: we have therefore

$$£88\tfrac{1}{2} \;:\; £1490 \;::\; £100$$

and the fourth term, or amount of stock bought by £1490 is £1683 12s. 3$\frac{141}{177}$d.

Ex. III. If a person invest £2000 in the Three per Cents. when they are at 95$\frac{1}{2}$, what is his annual income therefrom?

In this case the buyer gives £94$\frac{1}{2}$ for £100 stock, *i. e.* for the privilege of receiving £3 interest: hence the question is,—If £95$\frac{1}{2}$ produce £3 interest, what will £2000 produce? Our statement is

$$£95\tfrac{1}{2} \;:\; £2000 \;::\; £3$$

and the fourth term, or interest of £2000, will be found to be £62 16s. 6$\frac{102}{191}$d.

Ex. IV. Find what per centage will be obtained by investing in the Three and a-half per Cents. at 91: or in other words,—If £91 give £3$\frac{1}{2}$, what will £100 produce? The statement is

$$£91 \;:\; £100 \;::\; £3\tfrac{1}{2}$$

and the fourth term $= \dfrac{3\frac{1}{2} \times 100}{91}£ = \dfrac{\cancel{7}}{\cancel{2}} \times \dfrac{\overset{50}{\cancel{100}}}{\underset{13}{\cancel{91}}}£ = £3\frac{11}{13}$. Also transferring the 100 from the left-hand numerator to the right-hand denominator, I have

$$\frac{3\frac{1}{2}}{91} = \frac{3\frac{11}{13}}{100}.$$

Observing this equation, I notice that the right-hand side gives the ratio of the interest of £100 to £100; and the left-hand side gives the ratio of the interest on £100 *stock* to the price of that stock. Now, in finding the per centage which any interest produces, we wish to know what is the ratio of the interest on £100 to £100; and since the right-hand fraction gives this ratio, therefore the former fraction also gives it: hence this fraction gives a standard, by which we can compare the value of the per centages derived from any two investments in different kinds of stock.

Thus, if I wish to know whether it will be more advantageous to invest in the Four per Cents. at 95, or in the Three per Cents. at 85, I must compare the fractions $\frac{4}{95}$ and $\frac{3}{85}$.

By (46), the difference is $\frac{4 \times 85 - 3 \times 95}{95 \times 85} = \frac{340 - 285}{95 \times 85}$. Hence, since $340 > 285$, therefore $\frac{4}{95} > \frac{3}{85}$; or an investment in the Four per Cents. at 95 will produce better interest than in the Three per Cents. at 85.

If we wish to know how much better interest is obtained in the one case than in the other, we must observe that these fractions $\frac{4}{95}$ and $\frac{3}{85}$ express the respective portions of £100 which the two investments give as interest per cent:

hence, the difference of the per centages obtained $= \frac{340 - 285}{95 \times 85}$ of £100

$$= \frac{\overset{11}{\cancel{55}} \times \overset{20}{\cancel{100}}}{\underset{19}{\cancel{95}} \times \underset{17}{\cancel{85}}}£ = \frac{220}{323}£ = 13\text{s}\ \ 7\text{[illegible]d.}$$

162. We will now give an Ex. combining two or more of the operations exhibited in the previous Exs.

Ex. V. A person transfers £1000 stock from the Four per Cents. at 90 to the Three per Cents. at 72; find how much of the latter stock he will hold, and the alteration made in his annual income.

The first part of the question may be thus expressed: "If a certain sum of money will buy £1000 stock at 90, how much can be bought when the stock is at 72?" The statement will be

$$£72 \ : \ £90 \ :: \ £1000$$

and the fourth term $= \frac{\overset{125}{\cancel{1000}} \times \overset{10}{\cancel{90}}}{\underset{8}{\cancel{72}}}£ = 1250£.$

To find the income derived from the £1000 stock and from the £1250 stock, two simple statements might be employed: but where, as in this case, the stock consists of £100 shares, we can work more briefly thus:— £1000 stock = 10 cents.; and since each cent. produces £4, the whole 10 produce £40. Also, £1250 stock = $12\frac{1}{2}$ cents.; and since each cent. of this stock produces £3, the whole produce $12\frac{1}{2} \times 3£ = \frac{25}{2} \times 3£ = \frac{75}{2}£$ = £37 10s.; hence the difference of the incomes from the two investments = £40 − £37 10s. = £2 10s.

Exs. 56.

1. What must be given for £2000 Stock when the funds are at 85?

2. When $3\frac{1}{2}$ per cent. Stock is at $93\frac{1}{2}$, what sum will buy £1250, allowing $\frac{1}{8}$ per cent. for brokerage?

3. How much Stock can be bought for £1176 10s. when the funds are at £$90\frac{5}{8}$, and broker's charge 2s. 6d. per cent.?

4. What sum must be invested in 3 per cent. Stock at $94\frac{1}{2}$, to yield an annual income of £500?

5. If a person invest in the 3 per cents. at 93, at what rate per cent. will he receive interest for his money?

6. A person lays out £1000 in $3\frac{1}{2}$ per eent. Stock, when the funds are at $92\frac{5}{8}$; what income does he derive from it?

7. A sum of £999 19s. $11\frac{1}{4}$d. in the $3\frac{1}{2}$ per cents. produces £44 0s. 6d.; what was the price of the Stock when the money was invested?

8. In what Stock must I purchase, so that I may derive an income of £75 from the investment of £1875 at par?

9. A capitalist invests for a short period £100,000 in 3 per cents. at $87\frac{1}{2}$: when he sells out, they have risen 2 per cent.; what does he gain, reckoning $\frac{1}{8}$ per cent. for brokerage, both in buying and selling?

10. What must be the price of a railway share, paying a 5 per cent. dividend, and of which the nominal value is £100, so that a purchaser may receive 7 per cent. for the money invested?

11. A railway share, originally costing £100, has paid a dividend of 8 per cent.: what must I give for such a share, so as to receive $4\frac{1}{2}$ per cent. for my money?

12. Railway shares which were purchased at a discount of $10\frac{3}{8}$ per cent., and sold at a premium of £$31\frac{1}{8}$, realised a profit of £357 2s. $10\frac{2}{7}$d.: how much was invested?

13. A person having an annual income which arises from £450 invested in the 3 per cents., exchanges it for an annual income arising from £315 in the 4 per cents. Stock: what is his annual gain or loss by the exchange?

14. What is the interest for $5\frac{3}{8}$ yrs. of £293 invested in the 3 per cents., when they were at $87\frac{7}{8}$?

15. A person sells out of the $3\frac{1}{2}$ per cents. at $93\frac{1}{2}$, and realizes £18700: if he invest one-fifth of the produce in the 4 per cents. at 96, and the remainder in the 3 per cents at 90, find the alteration in his income.

16. Which is the better investment, to buy in the 3 per cents. at 85, or the 4 per cents. at 102; and by how much?

17. Find the difference of income arising from two investments of £5000: (1) in shares at $131\frac{1}{8}$, paying a 6 per cent. dividend; (2) in Bank Stock at $194\frac{1}{4}$, paying an 8 per cent. dividend.

18. A person wishes to bequeath an annuity of £100 a year; what sum must he devote to the purchase of $3\frac{1}{2}$ per cent. Stock at 97, so that the annuitant may receive the £100 free of 3 per cent. income tax?

EQUATION OF PAYMENTS.

163. If a person owe another several sums to be paid at different times, and it is required to know at what time it would be just to pay the whole at one payment, this would be a question to be solved by a Rule called Equation of Payments.

To ascertain the method of finding this time of payment, called the equated time, let us take a simple Example.

Ex. I. If £100 be due at the end of six months, and £200 at the end of twelve months; find when it is just to pay the whole in one sum.

It is quite clear that the time of payment will be at some period between the two fixed times, six months and twelve months; hence the former sum, £100, will be paid after it is due, and the latter sum, £200, before it is due.

Now, a person keeping the £100 beyond the appointed time ought, of course, if that were the only money to be paid, to pay interest for it; but, instead of paying interest, he is to make up for the privilege of keeping the £100 by paying the £200 before it is due; it is hence quite clear that he must pay this £200 such a time before it is due, that the interest of the £200 for that time shall just balance the interest he might obtain by keeping the £100 after it was due. The question then really is—How soon will the interest upon £200 produce the same as the interest upon £100? The answer evidently is, in half the time; *i.e.* the time of paying the £200 must be *earlier* than its original time of twelve months, by half as much as the time of paying the £100 is *later* than its original time of six months; therefore, if the payer keep the £100 *four* months beyond the six months, and pay the £200 *two* months sooner than the twelve months, the interest gained in the one case and lost in the other will be just balanced; and the whole sum will have to be paid in ten months.

164. Putting the question in another form, we may consider that when the sums were to be paid at different times, the payer had the £100 in his hands six months, *i. e.* he had the interest of £100 for six months, or of £600 *for* one month: also, he had the interest of £200 for

twelve months, or of £2400 for one month; therefore he had in all the interest of £2400 + £600, or of £3000 for one month. If, then, the debtor has to pay the £300 in one sum, how long ought he to keep it, so that its interest shall equal the interest of £3000 for one month? The answer to this question will be obtained from the following statement

£300 : £3000 :: 1 month;

and the fourth term will be ten months, the same result as before.

165. The truth of the above method has been disputed by arithmeticians upon this ground. Referring to the above Ex., it is said, that since the £100 is paid after it is due, the payer should pay the interest for the time that it is kept back; but since the £200 is paid before it is due, discount only should be allowed thereon: and since the discount is less than the interest, it is said that the payer, by the above method, receives as much more than his due as the interest of £200 exceeds the discount; hence, to make the payment perfectly correct according to this view, we ought to place the payment earlier than ten months. But since we have shown, by the working of the Ex. given above, that the interest gained in one case and lost in the other is equal, we shall, by placing the time for payment of the £300 earlier, rob the payer in two ways—1st, by depriving him of the interest of the £100 during the latter part of the four months that he ought to be allowed to hold it: and 2ndly, by making him put the £200 into the hands of his creditor for a longer time than two months.

The fallacy of the reasoning which would place discount instead of interest in the question may be shown thus. It is here assumed that discount, not interest, is applicable to

all cases where money is to be paid before it is due. Now, this is quite true where only one payment is to be made; for in that case we have laid it down, that the debtor is to pay such a sum as put out to interest shall just amount to the sum due: therefore, in all cases of discount the debtor pays less than he owes. But this is not the case in Equation of Payments: for here the creditor receives the whole of the latter portion of the debt, say £200, *though it be before it is due,* and he can put the *whole* out to interest, which he cannot, in real questions of discount; and the question now is,—not how much must be put out to interest to raise £200,—but how long must the whole £200 be put out, to raise an amount of interest equal to that which the creditor has lost, by allowing the first payment, say £100, to remain in the debtor's hands beyond its time. The question considered in this view has been satisfactorily answered in the Ex. worked above.*

Ex. II. A person owes £800; £200 to be paid in three months, £100 in four months, £300 in 5 months, and £200 in six months: if the whole were to be paid at once, what would the time of payment be?

Here, by the agreement, the debtor has the interest of

	£					£			
	200	for	3	months,	or	600	for	1	month
also of	100	„	4	„	or	400	„	1	„
also of	300	„	5	„	or	1500	„	1	„
lastly, of	200	„	6	„	or	1200	„	1	„
	800					3700			

therefore he has altogether the interest of £3700 for one month; and the whole sum to be paid is £800: hence, we have to find in how many months £800 will produce as much interest as £3700 in one month. The result is obtained from the following statement:

$$£800 : £3700 :: 1 \text{ month};$$

and the fourth term $= \frac{3700}{800} \times 1 \text{ month} = 4\frac{5}{8}$ months.

*See Art. *Rebate* in the "Penny Cyclopædia."

In the statements which are used in Exs. I. and II. we observe that the third term is one month; hence, since this will in every Ex. be the third term, we shall merely have to divide the second term by the first, in order to obtain the fourth term: also, the first term is the whole amount due; and the second is the amount of all the several sums, when each one is multiplied by its time of payment. Hence the equated time is found by multiplying each sum into its specified time, and dividing the total of the products so found by the whole amount to be paid

Exs. 57.

1. Find the equated time of payment of £160, whereof £100 is due in 3 months, and £60 in 8 months.
2. I owe £100 in 2 yrs., £230 in $2\frac{1}{2}$ yrs., and £280 in 3 yrs. hence: when must I pay the whole in one sum, so as neither to gain nor lose?
3. Of a debt, $\frac{1}{3}$ is due in 4 months, $\frac{1}{4}$ in 6 months, $\frac{1}{5}$ in 8 months, and the remainder in 12 months; find the equated time of payment.
4. Find the equated time of paying £200, which is due in monthly instalments of £20.

EXCHANGE.

166. Exchange, in its simplest meaning, is merely the conversion of any sum of money from the coinage of one country to that of another. To perform this conversion, we must of course know for how much a certain coin of one country can be exchanged in coins of another: for instance, if I wish to exchange a sum of English money for French, I must know how many of the current coins of France, viz. *francs*, will be given me for £1 of English money. This rate of Exchange between two countries is called the *Course of Exchange*, but it is not always the same; and we shall presently show the cause of the variation.

Ex. I. Exchange £850 sterling for francs at 25 francs 15 centimes, or cents; or, in other words,—If £1 sterling can be exchanged for 25 francs 15 cents, what number of francs can be obtained for £850? To obtain this number, I have the following statement:

£1 : £850 :: 25 francs 15 cents.

Also, since 100 cents. make one franc, therefore cents may be expressed as decimal parts of a franc; *i. e.* 25 francs 15 cents = 25·15 francs;

$$\text{hence, the fourth term} = \frac{£850 \times 25{\cdot}15}{£1} \text{ francs}$$
$$= (850 \times 25{\cdot}15) \text{ francs}$$
$$= 21377{\cdot}5 \text{ francs}$$

or 21377 francs 50 centimes.

Ex. II. How many pounds sterling can be obtained for 8457 marks $15\frac{1}{2}$ schillings, Hamburgh, at the rate of 13 marks 12 schillings for £1 sterling? The statement is

13 mks. 12 sch. : 8457 mks. $15\frac{1}{2}$ sch. :: £1.

By Tables of Hamburgh Coinage, we find that 16 schillings = 1 mark: hence, reducing the first and second terms of the above statement to half schillings, we obtain by the usual process the fourth term = £615 2s. 6d.

167. The method of these two Exs. will enable us to convert any sums from the currency of one country to that of another, when the course of exchange between the two countries is known. But we must also be able to perform this conversion between two foreign countries.

Ex. III. Change 1932 florins at Amsterdam for ducats at Naples, the course of Exchange being $80\frac{1}{2}$ florins for 40 ducats. The statement is

$80\frac{1}{2}$ florins : 1932 florins :: 40 ducats.

and the fourth term will be found to be 960 ducats.

In the following Ex. the process is not so simple:

Ex. IV. Exchange 1000 American dollars for sterling money when Eng. money bears a premium of 10 p. c. in America.

Here, £110 worth of dollars in America would produce only £100 of English money—hence 1000 dollars must be reduced in the ratio of 110 to 100. Thus,

$$110 : 100 :: \overset{\text{dols.}}{1000} : \frac{1000}{11} \text{ dols.}$$

or $909\frac{1}{11}$ dols.

We have now to convert $909\frac{1}{11}$ dols., each 4s. 6d., into pounds sterling.

The required number of pounds sterling

$$= \frac{1000}{11} \times \frac{4\frac{1}{2}\text{sh.}}{20\text{sh.}} = \frac{9000}{44} = 204\tfrac{6}{11}$$

or £204 10s. $10\frac{10}{11}$d.

If I had wished to change Eng. money to American, I should have *increased* the number of English pounds in the ratio of 100 to 109.

We now proceed to explain a further and more comprehensive meaning of the term Exchange.

In commercial transactions between different countries it is not usual to pay for goods imported, in coin, or as it is sometimes called, in *specie* or *bullion:* and for two reasons—first, the quantity of foreign goods imported by a country like England is so great, that if paid for in coin, the payment would speedily drain all the coin out of the country, and business could not be carried on. Secondly, there would be the probable loss of the coin by wreck or otherwise in the transmission; besides that there would arise a loss of interest on the money while it was being sent to its destination.

We shall now show how these difficulties may be avoided, when we are dealing with countries which send goods as well as receive them, *i. e.* which export as well as import.

168. The following is a simple Ex. of the manner of conducting these transactions.

Suppose A and B to represent two merchants in America, and C and D two others in England: let C buy of A a thousand pounds worth of goods, and therefore owe him £1000; so, also, let B owe D £1000 by a similar purchase; then if these sums be paid in coin, £1000 must cross the Atlantic twice.

A	B
AMERICA.	
ENGLAND.	
C	D

But,. since C has to pay £1000 to A, he would as readily pay it to D in England, if by such payment he could get rid of his liability to A: so also B would pay A, if he could be rid of his debt to D. This simple transaction might, therefore, be completed thus:

Let C send to A a bill acknowledging the debt of £1000, and promising to pay £1000 to any one in England who may present the bill to him at the expiration of a certain time: A then sells to B this bill, and receiving £1000 for it, has no longer any claim upon C. B now sends this bill to D, and D uses it as a bill for £1000, until the expiration of the time named on the bill, when the money is paid by C. Thus these four merchants have been able to have commercial dealings to the amount of £1000 each, without any coin having left either country.

Of course the value of this bill for £1000 depends entirely upon the ability of C to *meet* it, that is, to pay the money at the expiration of the time agreed upon in the bill: and we often find that C, who was considered able to pay at the time he gave the bill, has become a bankrupt before the time of payment; hence the loss falls on D. And this explains the reason why, in a commercial country like England, the failure of one merchant, or firm, causes others to fail: for, in the case above, D may also be liable for bills as well as C; and if not able to obtain the money which he expected from C, may himself become a bankrupt; and so in turn cause other merchants the same loss which he is himself suffering from the failure of C.

Now, there are thousands of merchants in the situation of A and B in America, and similarly of C and D in England. Hence there are, as a general rule, merchants wanting to buy bills, and others wanting to sell them,

in both countries: and what has been said concerning England and America is true with respect to any two countries which export to and import from one another.

These purchases and sales of bills are, for the reason mentioned in (160), conducted through the medium of Bill-brokers.

169. We have just now been supposing that the two countries have imported and exported goods to an equal amount from one another. But suppose that goods had been sent from America to England to a greater amount than from England to America, for instance, to the extent of £1,000,000, then more bills to that amount would go from England to America than from America to England. The merchants in America have in this case plenty of bills, and of course want money for them: but as there are more persons wishing to sell bills than to buy, therefore the bills fetch a lower price: on the contrary, as bills in England are not so plentiful, there are more buyers than sellers, and the bills fetch a higher price than usual. When this increased price exceeds the cost of insurance and loss of interest upon coin sent over to America, the English merchant prefers sending coin instead of paying the increased rate for a bill: and by successive exportations of bullion, the balance of £1,000,000, which was against us, will be paid off: but this increased price of a bill, (which swallows up the profits of business,) or the alternative of paying in coin, causes merchants to be slow in importing until our exports have increased and helped to restore the balance. Here, also, whatever has been said concerning England and America, is of course equally applicable to any two countries which have commercial transactions with each other. In those countries, as South America, where gold and silver

are amongst their principal productions, bullion is as much a regular article of export, as woollen or cotton goods would be from England.

170. If the price of a bill in England, entitling the holder to receive gold in a foreign country, be less than the usual course of exchange, the exchange is said to be in favour of England.

DEF. The standard rate of exchange between any two countries is termed the *Par of Exchange*, or the *Arbitration Price:* but, as alluded to in (166), is not always the same as the *Course* of Exchange. Also, a *Bill on London* means a paper entitling the holder to obtain gold in London, to the value of the amount mentioned in the bill.

Arbitration is called *Simple*, or *Compound*, according as there are three or more places concerned.

Ex. V. Bills on Amsterdam, bought in London at 12 florins 15 cents per £ sterling, are sold in Paris at $57\frac{1}{2}$ florins for 120 francs: what is the rate of Exchange between London and Paris?

My object here is to express £1 in terms of francs. Working fractionally, I have

$$£1 = 12 \text{ florins } 15 \text{ cents}$$

$$\text{also, } 57\tfrac{1}{2} \text{ flor.} = 120 \text{ francs}$$

$$\text{therefore } 1 \text{ flor.} = \frac{120}{57\frac{1}{2}} \text{ francs}$$

and therefore £1 = 12 flor. 15 cents

$$= 12{\cdot}15 \text{ florins} = \frac{12{\cdot}15 \times 120}{57\frac{1}{2}} \text{ francs}$$

The process may also be represented as follows:

$$£1 = 12{\cdot}15 \text{ florins} \qquad \text{(E)}$$

$$\text{and } 57\tfrac{1}{2} \text{ flor.} = 120 \text{ francs} \qquad \text{(F)}$$

therefore, taking the product of the two quantities in the left-hand of (E) and (F), and the corresponding product of the two on the right-hand, we have

$$£1 \times 57\tfrac{1}{2} \text{ flor.} = 12{\cdot}15 \text{ flor.} \times 120 \text{ francs};$$

or, dividing both sides by $57\frac{1}{2}$ flor.

$$£1 = \frac{12{\cdot}15 \text{ flor.} \times 120 \text{ francs}}{57\frac{1}{2} \text{ flor.}}$$

$$= \frac{12{\cdot}15 \times 120}{57\frac{1}{2}} \text{ francs, by (67)}$$

$$= 25 \text{ francs } 35\tfrac{15}{23} \text{ cents.}$$

Observing equations (E) and (F), I notice that I commence (E) by placing on the left-hand side the standard coin of one of the two countries the arbitrated price between which I wish to find: also the left-hand side of (F) is expressed in the same coin as the right-hand of (E); and in like manner we must proceed with any number of equations, till the right-hand side of the last will be expressed in the coin of the second of the two countries which we wish to connect, as in the following question.

Ex. VI. A bill upon Hamburgh is bought at 13 marks $10\frac{1}{4}$ schillings per £ sterling, then sold at Amsterdam at $35\frac{3}{4}$ florins per 40 marks: if the proceeds are then remitted to Paris in French bills at $57\frac{1}{4}$ florins per 120 francs, what rate of exchange is there between London and Paris?

$$£1 = 13 \text{ m. } 10\tfrac{1}{4} \text{ sch.} = 13\frac{10\frac{1}{4}}{16} \text{ marks} = 13\tfrac{41}{64} \text{ m.}$$

$$40 \text{ m.} = 35\tfrac{3}{4} \text{ flor.}$$

$$57\tfrac{1}{4} \text{ flor.} = 120 \text{ francs}$$

therefore $£1 \times 40 \text{ marks} \times 57\frac{1}{4} \text{ flor.} = 13\frac{41}{64} \text{ m.} \times 35\frac{3}{4} \text{ flor.} \times 120 \text{ francs}$; and dividing both sides of the equation by the product, $40 \text{ m.} \times 57\frac{1}{4}$ flor., we have

$$£1 = \frac{13\frac{41}{64} \text{ m.} \times 35\frac{3}{4} \text{ flor.} \times 120 \text{ francs.}}{40 \text{ m.} \times 57\frac{1}{4} \text{ flor.}}$$

$$= \frac{13\frac{41}{64} \times 35\frac{3}{4} \times 120}{40 \times 57\frac{1}{4}} \text{ francs, by (67)}$$

$$= 25 \text{ francs } 55\tfrac{1405}{3664} \text{ cents.*}$$

We will just give an Ex. of the form in which the state of the exchanges between England and other countries is generally reported in the public prints. The following is from the *Times* of March 22nd, 1848, under the head "Money Market and City Intelligence."

---

*The process by which a connection is established between the first and last terms, as for instance, between pounds sterling and francs, is sometimes termed the *Chain Rule*, because these extremities are joined, as it were, by successive links. And the results which we have just investigated, might have been written down at once, by taking as the left-hand side of the equation, the coin whose equivalent is desired,—and for the right-hand side, a frn, of which the numr consists of the product of all the successive denominations contained in the question,—and the denr, of all the remaining coins of the same kind as the numr, with the exception of the last or required coin.

"The last quotation of gold at Paris was about 30 per mille premium, which would give an exchange of 25·91. The last quotation of short bills on London being 26·50, the price of gold would appear to be about 2¼ per cent. higher in London than in Paris."

Here a comparison is instituted between the *nominal* rate of exchange of £1 for francs, and the *real* rate at the particular time mentioned. Thus, if 1000 represent the price of gold in Paris when £1 sterling is worth about 25·15 francs, what will be the value of this £1 in francs when 1030 represents the price of gold in Paris? We have, of course, this statement:

$$1000 : 1030 :: 25{\cdot}15 \text{ francs}$$

$$\text{the fourth term} = 25{\cdot}90 \text{ francs}$$

that is, £1 will in Paris produce 25·90 francs. But a person in Paris wishing to buy a bill on London, entitling him to receive £1 in gold, must give 26·50 francs. The difference, ·6 francs, is the amount by which £1 sterling in London is dearer than in Paris.

$$\text{Also, since } 26{\cdot}50 \text{ francs} = £1,$$

$$\text{therefore 1 franc} = \frac{1}{26{\cdot}5}£;$$

$$\text{and this extra price of } £1 = {\cdot}6 \text{ fr.} = {\cdot}6 \times \frac{1}{26{\cdot}5}£;$$

$$\text{and therefore the extra price of } £100 = 100 \times {\cdot}6 \times \frac{1}{26{\cdot}5}£.$$

$$= \frac{60}{26{\cdot}5}£ = \frac{12}{5{\cdot}3}£ = 2{\cdot}26\ldots.£$$

$$\text{or } 2\tfrac{1}{4}£ \text{ nearly.}$$

That is, the price of gold in London is 2¼ per cent. greater than in Paris; or a bill which would entitle a person to receive £100 in gold in London would cost £102¼ in Paris.

Exs. 58.

FRANCE AND ENGLAND.

The course of Exchange between France and England is 25 fr. 30 cents for £1 sterling. 100 cents = 1 franc.

1. Exchange £350 for francs, at 25 fr. 50 cts. per £ sterling.
2. " £75 10s. " 25·35½ "
3. " £425 15s. 6d. 26·5 "
4. " 9349·90 francs for pounds, at 25·90 per £ sterling
5. " 21475·12 " 25·75 "
6. " 1875·5 " 26·5 "

HAMBURGH AND ENGLAND.

13 marks 8 schillings = £1 English. 16 schillings = 1 mark.

			m.	sch.	
Exchange	£425	for marks, at	13	12	for £1.
„	£375 10s.	„	13	9½	„
„	£87½	„	13	8	„

			m.	sch.	
„	1000 marks	for pounds sterling, at	13	8	for £1.
„	8754 mks. 15 sch.	„	13	10½	„
„	3537·45 mks.	„	13	11	„

THE UNITED STATES AND BRITISH NORTH AMERICA.

llar = 4s. 6d.; also sterling money generally bears a premium of 9 per cent.;
e. £100 sterling can be exchanged for as many dollars as will amount to £109.

Exchange	£100	for dollars when Eng. money has prem. of		8 p. c.
„	£425 10s.	„	„	10 „
„	£1256 5s.	„	„	9½ „
„	1000 dollars	for Eng. money, when the prem. is		8 „
„	3225	„	„	7½ „

EAST INDIES. 1 Rupee = about 23¾d.

„ £500 for rupees, at 23½d. per rupee.

„ 7500 rupees for sterling money, at 24½d.

). Exchange 2420 rupees for francs, the course of exchange being upees for 94 francs.

l. Change 1750 marks of Hamburgh for florins at Amsterdam, at rate of 135 marks for 120 florins.

2. Exchange 3700 francs for Hamburgh marks, at the rate of 187½ cs for 100 marks.

ARBITRATIONS OF EXCHANGE.

3. If the Exchange between London and Amsterdam be 11¾ florins £ sterling, and between Amsterdam and Paris be at the rate of 55½ ns per 115 francs; find the rate of exchange between London and s.

4. Bills on Paris, bought at the rate of 25 francs 35 cents per £ ing, are sold in Lisbon at 190 rees per franc; what rate of exchange ere between London and Lisbon?

5. The exchange between London and Hamburg is 13 mrks. 10 sch. £ sterling; between Hamburgh and Amsterdam is 150 marks for 275 cs; what exchange does that give between London and Paris?

26. If the exchange between Amsterdam and Hamburgh be at $11\frac{3}{4}$ flors. for $13\frac{1}{2}$ marks, between Amsterdam and Genoa be at 120 flors. for 25 lire, between Genoa and Portugal be 5 lire for 800 rees, between Lisbon and London be 1 milree or 1000 rees for 54d., what exchange does this give between London and Hamburgh? and what difference will there be, between remitting £500 from London to Hamburgh by this circular route, and sending it direct, at an exchange of 13 mrks. 11 schgs. per £ sterling?

27. The premium on gold at Paris is $5\frac{1}{4}$ per mille, which gives an exchange of 25·29; if the quoted exchange at Paris on London be $25·22\frac{1}{2}$, shew that gold is 0·26 per cent. dearer in Paris than in London.

Exs. 59. M.

1. A debt of £144 7s. 6d. was paid in an equal number of guineas, half-guineas, and seven shilling pieces: required the number.

2. A peck of flour gives 20lbs. of bread; how much land would grow corn enough for $1\frac{1}{4}$ millions of people for a week, at $4\frac{1}{2}$ quarters to the acre, and $1\frac{1}{4}$lbs. of bread for each person per day?

3. The population of a town rises 1 per cent. for 3 years successively; if, at the beginning of the 3 yrs. the population were one million, what would it be at the end?

4. Tea bought at 1s. $10\frac{1}{2}$d. per lb. pays a duty of 2s. $2\frac{1}{4}$d. per lb., what per centage of the whole cost is the taxation?

5. The $3\frac{1}{2}$ per cents. are reduced to $3\frac{1}{4}$ per cents.; 150 millions of stock are so converted: but the holders of 6 millions dissent; if they are paid off while the stock is at $97\frac{1}{2}$, how much will the nation gain or lose in the first year?

6. An estate of 270 acres is bequeathed to three tenants, to be divided in proportion to their rents, which are £180, £120, and £60; how must the land be divided?

7. Find a fourth proportional to $\frac{2}{3}$, $\frac{3}{4}$, $\frac{11}{10}$; and to ·05, ·015, and ·0075.

8. Find the difference between the Simple and Compound Interest of £520 for 2 yrs., at 5 p. c., where the interest is paid half-yearly.

9. *A*, *B*, & *C* enter into partnership; *A* puts in £100 for 6 months, *B* puts in £200 for 4 months, and *C* £100 for 15 months; divide a profit of £150 equitably.

10. At what rate must I sell an article which cost 50s., so as to gain 15 per cent.?

11. Find the purchase money of £1500 stock in the 3 per cents., at $88\frac{1}{2}$, including $\frac{1}{8}$ per cent. commission.

12. At what rate per cent. will £100 double itself in 8 years, S. Int.?

13. Find the annual income from a legacy of £5000 Stock in the $3\frac{1}{2}$ per cents., after paying the legacy duty of 10 per cent.

14. Prove that the sum of the fractions $2\frac{1}{12}$ and $\dfrac{2}{1\frac{4}{9}}$ is equal to 5 *times* their difference.

N.

1. Shew by a simple Ex., that Mult^n is nothing more than a shortened mode of Addition.

2. What would be the length of an acre of ground, if its breadth were 40 yards?

3. Find the value of $4{\cdot}05 \times {\cdot}000012$; also of $4{\cdot}05 \div {\cdot}00012$; and prove both results by vulgar fractions.

4. What does £25 15s. amount to, when taken ·125 times?

5. If the means of a proportion be 9 and $1\dot{6}$, and one of the extremes be 56, what is the other extreme?

6. A person bequeathed £5000 to be divided amongst three persons, in the proportions of 3, 5, and 7; find their respective shares.

7. Explain what fractions produce terminating, and what produce non-terminating decimals. Give Exs. of each.

8. Find the exact difference between ·127s. and $\cdot 12\dot{7}$s.; and express the result as the fraction of a crown.

9. What sum of money will in 5 years amount to £411 5s. at $3\frac{1}{2}$ p. c. Simple Interest?

10. Find the amount of insurance upon £12500, at 4s. 6d. per cent.

11. What is the present value of £463 10s. due in 8 months, allowing $4\frac{1}{2}$ per cent?

12. How many yards at 6s. $7\frac{1}{2}$d. per yd. must be given in exchange for 105 yds. at 3s. 4d. and 375 at 4s. $10\frac{1}{2}$d.?

13. Explain the meaning of the terms *directly* and *inversely* proportional; and give an Ex. of a question illustrating each expression.

14. What decimal multiplied by $\frac{2}{3}$ of $\frac{7}{8}$ of $3{\cdot}\dot{3}$ will become 17?

15. How many years' purchase should be paid for property, so as to receive $6\frac{3}{4}$ per cent?

O.

1. Find the Simple Interest of £500 for 5 years at $3\frac{1}{2}$ per cent.

2. What is the number of cubic yds. in 1438790 cubic inches?

3. How often must the sum of 2s. $6\frac{1}{4}$d., 3s. $9\frac{1}{2}$d. and 18s. $8\frac{1}{4}$d. be repeated, to make £100?

4. If 10 francs be worth 6 florins, and 75 florins be equivalent to 4 moidores; how many francs must be given for 16 moidores?

5. Find the frac^n which being multiplied by $\frac{3}{4}$ of $\frac{7}{9}$ of $2\frac{3}{11}$ gives a product = 1.

6. The interest on a railway share is $3\frac{3}{4}$ per cent.; what is the market value of the entire line, if money be worth 5 per cent., and the amount paid in dividends be £150,000?

7. A sum of money has doubled itself in 17 years at Simple Interest; what is the rate per cent.?

8. I sell an article for 15s. 9d., and by so doing gain 17 per cent.; what was its prime cost?

9. An estate which brings $3\frac{1}{2}$ per cent. lets for £546, what was the purchase money?

10. Divide £70 amongst three persons, whose shares shall be in the ratio of the numbers $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$.

11. Shew whether it is better to invest in the 3 per cents. at 89, or the $3\frac{1}{2}$ per cents. at 94: What difference would there be, if £10000 stock were held?

12. Compare as vulgar fractions $\cdot025 \times \cdot07$, $11\cdot035 \times \cdot0008$, and $\cdot1\dot{9} \times \cdot00\dot{3}$.

13. A person mixes 25 bushels of wheat at 4s. 9d., 36 at 5s. 6d., and 15 at 6s. 6d.; what must be the selling price per bushel of the mixture, to gain 10 per cent. on the above prices?

14. Exchange £1050 for francs, at 25 fr. 50 cts. per £ sterling.

15. Exchange 9062 fr. $62\frac{1}{2}$ cts. for pounds sterling, at 25 fr. 35 cts. for £1.

P.

1. Find a fourth proportional to 1, $\frac{1}{2}$, $\frac{1}{4}$; also a third proportional to 12 and 25.

2. Express $\dfrac{14\frac{3}{4}}{\cdot025}$ as a simple ratio.

3. I lend £175 for 6 months when money is worth 7 per cent.; for what time ought I to be able to borrow £250, when money is worth but $4\frac{1}{2}$ per cent?

4. Explain the difference between Interest and Discount; and find the true discount of a bill of £65 13s. at 4 months, drawn Oct. 4th, and discounted Nov. 26th, at 5 per cent.

5. At what rate of interest would £350 amount to £389 7s. 6d. in 3 yrs., at Simple Interest?

6. Goods bought at £2 4s. 6d. are sold at £2 18s. 9d.; required the profit per cent.

7. A railway share, originally costing £50, has paid a half-yearly dividend of £1 10s.; what will be my rate of interest, if the share cost me $£55\frac{1}{2}$?

8. What will be the first year's expense of an insurance on £1500 at a premium of £2 13s. 10d. per cent., and a stamp duty of £3? Find the per centage, including the stamp.

9. If the price of 500 bricks, of which the length, breadth and thickness are 12, $4\frac{1}{2}$, and 3 inches respectively, be 12s. 6d.; how many shall I obtain for the same money, if the dimensions be 15, 6, and 4 inches?

10. Find the prices of investment in the 3, $3\frac{1}{2}$, and 4 p.c. Stocks, when they each pay $3\frac{1}{4}$ per cent.

11. Distribute the sum of 1000 guineas in the ratio of 1, $\frac{1}{3}$, $\frac{1}{5}$.

12. The ratio of the invoice price to the net price is 11 : 8; what per centage has been thrown off as discount from the invoice price?

13. Exchange £415 10s. for dollars at 4s. 6d., when English money bears a premium of $7\frac{1}{2}$ p. c.

14. Exchange 3500 dollars for sterling money, when the premium on English money is $10\frac{1}{2}$ per cent.

16. The exchange between London and Paris is 25·5 francs per £ sterling; between Paris and Amsterdam is 117 francs for 55 florins; between Amsterdam and Hamburgh is 11 florins for 13 marks; what is the exchange between London and Hamburgh?

AREA AND VOLUME.

DUODECIMALS.

171. If any line be taken, as *a b*, and upon it a square, *a b c d*, be described, this figure may be called the square of *a b*. And if we take *a b* as an unit of length, viz. 1 inch, 1 foot, &c., then *a b c d* is called 1 square inch, 1 square foot, &c. We have here, therefore, inches, feet, &c. of length, or *linear* inches; and inches, &c. of surface, or *superficial* inches, or square inches.

FIG. 1.

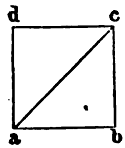
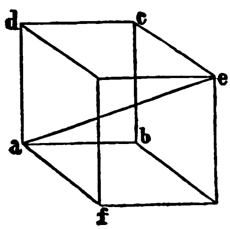


FIG. 2.



172. In like manner, if upon *a b*, the annexed figure *a b c d e f* be described, having six sides, or surfaces, each equal to *a b c d*, it is called a *cube*: and, as before, if *a b* be taken as representing 1 inch, 1 foot, &c., this figure will represent 1 cubic inch, 1 cubic foot, &c.

We have now, therefore, three kinds of units of measurement, viz. linear, or common inches; square, or superficial inches; and cubic, or solid inches. Also, these three units are said to contain 1, 2, and 3, dimensions respectively. For example, the floor of a room, having the dimensions length and breadth, is of the same nature as a square, and any such surface is called an *Area:* but a cistern of water, having the three dimensions of length, breadth, and depth, is of the same nature with a cube. The quantity contained by such a cistern or similar figure is termed its *Volume,* or solid content.

DEF. A surface which is so enclosed with lines, that any two which meet in a point are perpendicular to one another, is called *rectangular.*

OBS. Any surface bounded by straight lines may be denoted by two letters placed at its opposite corners; and any solid contained by such surfaces may be so denoted: but the two letters employed should not be joined together by a line. Thus, in Fig. 1, I should say the area *b d*, not *a c*; and in Fig. 2, I should say, the volume *f c*, not *a e.*

DEF. The lines *a c*, *a e*, are called *diagonals.*

173. I have now to explain how to find the area of surfaces and the volume of solids: but as I do not propose to enter upon mensuration generally, I shall merely treat of rectangular surfaces, as squares and oblongs; and of solids, the surfaces of which are also rectangular.

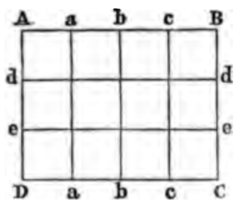
174. Quantities which can be accurately represented by numbers, whether whole or fractional, are called *commensurable;* and those which cannot be so represented, are called *incommensurable.* Thus it will be shown under the head of "Ratio," in the Appendix, that if in Fig. 1 the length of *a b* be represented by 1, *a c* cannot be accurately

represented by any number whatever, whether whole or mixed. The quantity used for the value of $a\,c$ is 1·4142..... and this being nearly the true value, is termed its *approximate* value.

175. We may here state as a fact, that the area of any rectangular surface is found by multiplying the numbers which represent its length and breadth; also that the volume of a solid, bounded by rectangular surfaces, is found by multiplying together the numbers representing the length, breadth, and depth or thickness. The above statements are true, whether the lines bounding the area or volume be commensurate or not; but they cannot be proved to be universally true without the aid of geometry. We shall, however, give an Ex. illustrating the correctness in each case, choosing of course only commensurable numbers.

176. Let $A\,B\,C\,D$ be a rectangular figure whose sides, $A\,B$ and $A\,D$, meeting in the point A, and called *adjacent* sides, contain an exact number of units,—viz. $AB = 4$ inches, $AD = 3$ inches; let the opposite sides, AB, CD, be divided into four equal parts in a, b, c, and the sides AB, CD, be divided into three equal parts, in d, e; let the lines $a\,a$, $b\,b$, $c\,c$, $d\,d$, $e\,e$, be drawn: the figure will be divided into equal squares, each of which has for its side 1 linear inch, as $A\,a$, and is therefore a square inch. Also, counting vertically, there are three rows of squares, because $A\,D = 3$ inches, and each row contains four squares, because $A\,B = 4$ inches: and there are in all 12 squares, *i.e.* 3×4 squares: hence we see that the

FIG. 3.



number of square inches in the area $ABCD =$ the product of the number of linear inches in the two adjacent sides, AB, AD.

Def. The lines $a\,a$, $b\,b$, which are drawn so that they are at an equal distance from one another, are called *parallel* lines; so also they are said to be *parallel* to AD or BC.

Hence, if I wished to draw lines, as $a\,a$, $b\,b$, or $c\,c$, in the above figure, I should say—through a, b, c, draw lines parallel to AD or BC. So also, $d\,d$, $e\,e$, are drawn parallel to AB, or CD.

If AD had been $= AB$, then the figure would have been a square, and the number of square inches in it would be 4×4, or $4^2 = 16$; *i. e.* the number of square inches in any square figure whose side is expressed in linear inches, is found by multiplying the number contained in the side by itself. Hence the second power of a number is called its square, because it represents the area of a square figure, the side of which is the number itself.

We can now show how the numbers mentioned in what is called "Square Measure," are obtained.

For if 12 linear inches = 1 linear foot,

therefore 12 in. × 12 in. = 144 square inches = 1 square foot,

So also, since 3 linear feet = 1 linear yard,

3 ft. × 3 ft. = 9 square feet = 1 square yard.

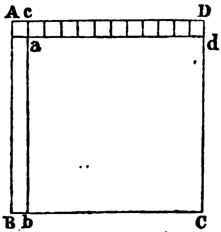
Again, $5\frac{1}{2}$ linear yards = 1 linear pole, or perch;

therefore $5\frac{1}{2}$ yds. × $5\frac{1}{2}$ yds. = $30\frac{1}{4}$ sq. yds. = 1 sq. perch.

and these square perches, yards, &c., are the quantities always made use of in measuring land; for no amount of perches in *length* could make up an acre, which consists of *surface*.

177. In measuring surfaces, such as square feet of timber, many arithmeticians have called the twelfth part of a square foot, as *Ad* or *Ab*, an *inch*, and a twelfth part of *A d*, viz. *A a*, a *part:* but since a pupil is taught in "Square Measure" that *A a* is an inch, and that 144 such inches make up a square foot; it is clearly absurd to call *Ad* an *inch*, seeing that 12 such make up a square foot: we shall, therefore, confine the name *inch* to *Aa*, or any quantity equal to it. But the divisions of *AC* have been, and may be, called superficial *primes, seconds, thirds*, &c.; where it is to be remembered, that a prime is $\frac{1}{12}$th of a square foot, and each succeeding denomination is $\frac{1}{12}$th of the one preceding it. So also, linear inches are sometimes called primes, and twelfths of an inch, seconds.

FIG. 4.



The square inch, *Aa*, might also be divided, exactly as we have divided *A C*; and its twelfth part would be called a *third*, and be similar in shape, though not in size, to *Ab* or *Ad*; and the twelfth part of this *third* would be called a *fourth*, and be similar in shape to *Aa*.

Also, observing Fig. 4, we learn that

1 foot × 1 foot = 1 sq. foot, as *A C*.

1 foot × 1 inch = 1 sup. prime, as *A b*.

1 inch × 1 inch = 1 sq. inch, or sup. second, as *A a*.

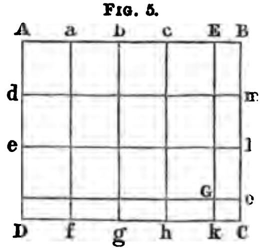
178. It must be distinctly noticed that we do not multiply together concrete quantities, as feet by feet, inches by inches, &c.; we merely multiply the *number* of feet or inches in the length by the *number* in the breadth; and then we observe that the number obtained in the product

is equal to the number of superficial units, primes, &c., in the area contained by the above length and breadth. The same remark applies to the multiplication of three dimensions, in finding the volume of a solid.

Attention to this fact, that concrete quantities cannot be multiplied together, will save persons from the absurdity of attempting to multiply pounds, shillings, or pence, by pounds, shillings, &c. I can multiply 5s. by the number 5, and the product is 25s.; but if I attempt to multiply 5s. by 5s., I know of no quantity which can correspond to such a product. And in considering (74) it must be noticed, that though in the practice of Rule of Three I appear to be multiplying the third term by the second, each of which is generally a concrete quantity, and then dividing by the first term, which is also generally a concrete quantity; yet, since the concrete multiplier and divisor are of the same kind, the result is that I have merely multiplied by a fraction, *i. e.* by an abstract number, generally fractional; and the common process has been, that I have multiplied the third term by the numerator, and divided by the denominator of this fraction.

179. We can now find the area of any rectangular figure contained by commensurable lines.

FIG. 5.



Let $ADCB$ be a rectangular figure, of which the side $AB = 4$ ft. 6 in., and the adjacent side $AD = 3$ ft. 3 in. Let AB be divided into feet in the points a, b, c, E, and AD so divided in the points d, e, F. Draw af, bg, ch, Ek, parallel to AD, or BC: and draw dm, el, Fo parallel to AB or DC, according to (176). Let Ek, Fo, intersect in G. Then the whole area $AC = AG + BG + DG + CG$.

Also, from (177) we have

		sq. ft.	sup. pr.	sq. in.
$AG = AE \times AF = 4$ ft. $\times 3$ ft. $= 12$ sq. ft.	$=$	12	0	0
$BG = EG \times EB = 6$ in. $\times 3$ ft. $= 18$ sup. pr.	$=$	1	6	0
$DG = FG \times FD = 4$ ft. $\times 3$ in. $= 12$ sup. pr.	$=$	1	0	0
$CG = Go \times Gk = 6$ in. $\times 3$ in. $= 18$ sq. in.	$=$	0	1	6
Area AC	$=$	14	7	6

180. Observing these four results, we may place the work in a condensed form, as annexed.

III.	II.	I.
	ft.	in.
	4	6
	3	3
1	1	6
13	6	
14	7	6

Since our proposal was to find the product of 4 ft. 6 in. by 3 ft. 3 in., we place one number under the other nearly as in Compound Multiplication, and commence multiplying the upper line by the lowest denomination in the lower line, and so on through the multiplier. I give the operations which are required to perform the work mentally, observing that each product as it is formed can be reduced to the next higher denomination by dividing it by 12. I commence at the right-hand and proceed thus: $3 \times 6 = 18$, which, divided by 12, gives 1 to carry to column (II.), and 6 to put down: $3 \times 4 = 12$ for column (II)., which, with the 1 carried, is 13, and divided by 12 gives 1 to carry to (III.), and 1 to set down in (II.) Again, commencing with the multiplier 3 feet, I have $3 \times 6 = 18$ in (II.), which divided by 12 gives 1 to carry to (III.), and a remainder 6 for (II.): lastly, $3 \times 4 = 12$ for (III.), which, with the 1 to carry, becomes 13. Adding the two rows, I have the result 14 sq. ft. 7 sup. primes, 6 sq. in.; or, bringing the primes to square inches, I have 14 sq. feet 90 sq. inches.

Comparing the operations of this article with those of the last, I notice that the steps which produced the first product of the multiplication sum in (180) are the same as the third and fourth in (179); and those which produced the second product are the same as the first and second in that article. Hence in Exs. similar to the one now worked, we need not draw any figure to insure the correctness of the work obtained by the multiplication in (180). This mode of working is called *Cross Multiplication*, and sometimes *Duodecimals*. The latter name is given in consequence of the work of such an Ex. being precisely the same as in Simple Multiplication; provided that, in working from right to left, we take every figure as having

twelve times the value which it would have one place to the right, instead of *ten* times that value, as in common numbers.

The above Ex. may be very briefly exhibited in a fractional form.

Thus, the area $= (4 \text{ ft. } 6 \text{ in.}) \times (3 \text{ ft. } 3 \text{ in.})$

$$= (4\tfrac{1}{2}) \text{ ft.} \times (3\tfrac{1}{4}) \text{ ft.},$$

$$= \frac{9}{2} \times \frac{13}{4} \text{ sq. ft.} = \frac{117}{8} \text{ sq. ft.},$$

$$= 14\tfrac{5}{8} \text{ sq. ft.} = 14 \text{ sq. ft. } 90 \text{ sq. inches,}$$

as before.

181. Next, let one or both of the adjacent sides of any rectangular figure, whose area is required, consist of feet, inches, and twelfths of an inch; we must then further divide a square inch as we divided a square foot in (177): and we then learn that

1 inch $\times \frac{1}{12}$ inch $= \frac{1}{12}$th of a sq. inch, or 1 super. third;

$\frac{1}{12}$ inch $\times \frac{1}{12}$ inch $= \frac{1}{144}$ super. inch, or 1 superficial fourth.

We can now, without further explanation, follow the work of the accompanying Ex.

Find the area of a rectangular floor, whereof the length is 9 feet 4 inches 7 seconds, and the breadth 5 feet 6 inches 4 seconds.

		ft.	pr.	sec.
		9	4	7
		5	6	4
	3	1	6	4
4	8	3	6	
46	10	11		
51	10	4	0	4

The highest denomination, 51, in this product has principally been obtained from the multiplication of 5 feet and 9 feet, and is therefore square feet; and the remaining denominations are superficial primes, seconds, thirds, fourths: and the whole answer is written 51 sq. feet 10 sup. pr. 4 sec. (or sq. in.) 0 thirds 4 fourths; or neglecting the fourths, 51 sq. feet, 124 sq. inches.

Exs. 60.

1. Multiply 4 ft. 7 in. by 8 ft. 3 in.
2. „ 13 ft. 5 in. by 27 ft. 9 in.
3. „ 2 ft. 6 in. 4 sec. by 11 ft. 3 in.
4. „ 18 ft. 4 in. by 3 ft. 6 in. 9 sec.
5. „ 7 ft. 2 in. 5 sec. by 11 ft. 3 in. 4 sec.
6. „ 15 ft. 0 in. 7 sec. by 13 ft. 0 in. 11 sec.
7. What is the cost of paving a rectangular area, 20 ft. 6 in. by 4 ft. 3 in., at 30s. per square yard?

8. A room is 97 ft. 8 in. in circuit, and 9 ft. 10 in. high; what will the painting come to at $7\frac{1}{2}$d. per square yard?

9. How much will remain out of 393 square feet of carpeting, after covering a floor 23 ft. 6 in. long, and 16 ft. 7 in. broad?

10. Find the whole surface of a box, whereof the height, length, and breadth are respectively 3 ft. 2 in., 7 ft. $5\frac{7}{12}$ in., and 4 ft. 8 in. 5 sec.

11. What is the whole surface of a cubical box, the edge of which is 2 ft. 9 in.?

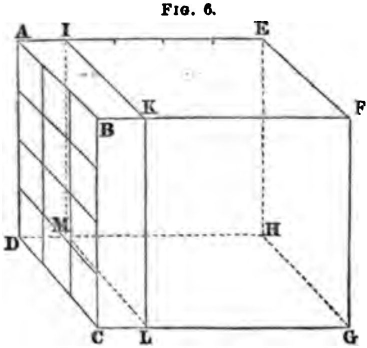
12. A house has 63 windows; 40 of them contain 12 panes, each 20 in. by 16 in.; the remainder contain 9 panes, 16 in. square; find the cost of glazing the whole at 2s. 3d. per square foot.

13. A Turkey carpet, measuring 11 ft. 6 in. by 9 ft. 8 in., is laid down on the floor of a room measuring 14 ft. by 12 ft. 6 in.; find the quantity of oil cloth necessary to complete the covering of the floor.

182. We have stated that the length, breadth, and depth of a solid must be multiplied together to obtain its volume; we now proceed to give an Ex. of the truth of this, just as we have already worked one in a question of two dimensions.

First, let all the dimensions consist of whole numbers, viz. 3, 4, 5 inches; and let $ABCGH$ be the solid where $AE = 5$, $AB = 3$, $AD = 4$.

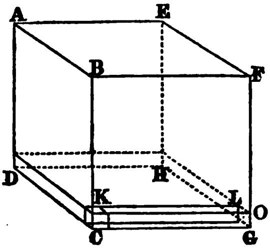
FIG. 6.



Let $IKLM$ be a section of the whole solid by a plane parallel to $ABCD$: then, since $AE = 5$, the whole solid may be divided by similar planes into solid blocks, the base of each of which is $ABCD$, and the thickness 1 inch: hence there will be five times as many solid inches in the whole AG, as there are solid inches in AL.

Now, by (176) the area $ABCD = 4$ in. $\times$ 3 in. $= 12$ sq. in: and if upon each of these square inches a solid inch be placed, there will be twelve solid inches in AL. Hence, in the whole AG, *i. e.* in five times AL, there will be 5×12 cubic inches; *i. e.* the volume of the solid $= 5 \times 4 \times 3$ solid inches. Of course, if the three dimensions had been expressed in feet instead of inches, the result would have been $5 \times 4 \times 3$ solid feet.

FIG. 7.



Let now the solid have all its dimensions equal, and each = 1 foot, then the figure is a cubic foot, and its volume = 12 in. × 12 in. × 12 in. = 1728 cub. in., and 12 × 12 × 12, or 12^3 is called 12 cubed. Hence we can obtain the numbers exhibited in what is called "Solid Measure."

For 12 in. × 12 in. × 12 in., or 1728 solid inches = 1 solid foot;

3 ft. × 3 ft. × 3 ft., or 27 solid feet = 1 solid yard.

By a classification similar to that of (177), we term a twelfth part of a solid foot, a solid or cubic prime; a twelfth of a prime, a second; and so on: but we must always remember, that though we use but one set of names, we have three kinds of primes, seconds, &c., viz. linear, superficial, and solid; and the various sorts can never be added to or subtracted from each other—for it is evidently impossible to add an area either to a line or a volume.

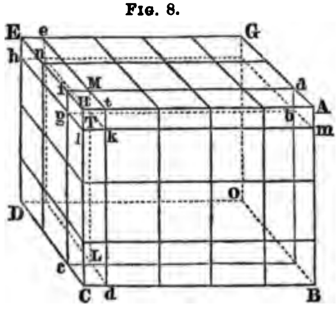
By observing Fig. 7 we learn that

1 sq. foot × 1 linear inch = 1 solid prime, as *DO*.

1 sup. prime × 1 linear inch = 1 solid second, as *CL*.

1 sq. inch or 1 sup. sec. × 1 linear inch = 1 solid third, or solid inch, as *CK*.

FIG. 8.



Let *ABCDEG* or *BE* be a solid contained by rectangular surfaces; let the three adjacent edges, *CB*, *CD*, *CH*, be respectively equal to 4ft. 5in., 2ft. 6in., and 3ft. 4in.; let the surfaces *AC*, *CE*, *EA*, be intersected by *parallel* lines drawn through the extremities of the feet in

their respective sides, as in (179); let M be the point in the surface AE in which the lines aMf, tMe, bounding the complete squares in AE, intersect; ml, lh, the lines bounding the squares in the surfaces AC, CE, intersect in l, a point in the edge CH.

Through	aMf	draw a plane	ac	parallel to	AC
„	eMt	„	ed	„	EC
„	mkl	„	hm	„	AE

The planes	ac	and	ed	shall intersect in the line	MTL
„	ac	„	hm	„	gTb
„	ed	„	hm	„	kTn

These three lines all pass through the point T; and, measuring from T, the figure may be divided into the following solids :—

(1) TO, TB, TC, TD, upon the bases LO, LB, LC, LD, and all having the same altitude LT.

(2) TG, TA, TH, TE, upon the bases MG, MA, MH, ME, (which are precisely the same as the bases of the other four,) and having an altitude MT.

We shall now find the volume of each of these solids according to (182); and show that a process of multiplication, similar to that in (180) will produce a result corresponding to the value actually obtained from the figure.

183. By (179) we may at once write, that

The area of $GH = EG \times GA =$ (4 ft. 5 in.) × (2 ft. 6 in.)
= 11 sq. ft. 0 pr. 6 sec. (I.)

and hence we shall have the volume of GC
= area $GH \times ML =$ (11 sq. ft. 0 pr. 6 sec.) × (3 ft. 4 in.) (II.)

By observing the four products in the multiplication of (I.) we find that they are respectively

sq. ft.	pr.	sec.		
8	0	0	=	area GM or OL
2	0	0	=	„ AM or BL
	10	0	=	„ EM or DL
		30	=	„ HM or CL
11	0	6	=	area GH or DB

Hence we may write the product (II.) in the following shape:

sq. ft. pr. sec.					sol. ft. pr. sec. thir.	sol. ft. pr. sec. thir.		
8 0 0	$\times \left\{ \begin{matrix} 3\text{ ft.} \\ 4\text{ in.} \end{matrix} \right. =$	$\left\{ \begin{matrix} OL \times TL \\ GM \times TM \end{matrix} \right\} =$	$\left. \begin{matrix} OT \\ GT \end{matrix} \right\} = LG =$		24 0 0 0 = 32 0 0 =	24 0 0 0 2 8 0 0	(B)	 (A)
2 0 0	$\times \left\{ \begin{matrix} 3\text{ ft.} \\ 4\text{ in.} \end{matrix} \right. =$	$\left\{ \begin{matrix} BL \times TL \\ AM \times TM \end{matrix} \right\} =$	$\left. \begin{matrix} BT \\ AT \end{matrix} \right\} = LA =$		6 0 0 0 = 8 0 0 =	6 0 0 0 8 0 0	(B)	 (A)
10 0	$\times \left\{ \begin{matrix} 3\text{ ft.} \\ 4\text{ in.} \end{matrix} \right. =$	$\left\{ \begin{matrix} DL \times TL \\ EM \times TM \end{matrix} \right\} =$	$\left. \begin{matrix} DT \\ ET \end{matrix} \right\} = LE =$		30 0 0 = 40 0 =	2 6 0 0 3 4 0	(B)	 (A)
30	$\times \left\{ \begin{matrix} 3\text{ ft.} \\ 4\text{ in.} \end{matrix} \right. =$	$\left\{ \begin{matrix} CL \times TL \\ HM \times TM \end{matrix} \right\} =$	$\left. \begin{matrix} CT \\ HT \end{matrix} \right\} = LH =$		90 0 = 120 =	7 6 0 10 0	(B)	 (A)
11 0 6	$\times \left\{ \begin{matrix} 3\text{ ft.} \\ 4\text{ in.} \end{matrix} \right. =$		BE		=	36 9 8 0		

Taking the area GH or DB, and multiplying it by 3 ft. 4 in. by cross multiplication, we have the annexed operation, where it will be found that the former of the two products equals the sum of all the products (A), and the latter equals the sum of all the products (B): and the result is 36 sol. ft. 9 pr. 8 sec.; or 36 sol. ft. 1392 sol. in.

sq. ft. pr. sec.	
11 4 6	
3 ft. 4 in.	
3 8 2 0	= (A) or vol. Gl
33 1 6	= (B) or vol. Ol
36 9 8 0	= vol. GC

The above value of the volume may be written

$$\left(36 + \frac{9}{12} + \frac{8}{144}\right) \text{ feet, or } 36 + \frac{108+8}{144}, \text{ or } 36\tfrac{116}{144}, \text{ or } 36\tfrac{29}{36} \text{ feet.}$$

and this result might,* as in (180) have been obtained at once, by expressing the three dimensions as fractional parts of feet, and finding their product:

Thus, (4 ft. 5 in.) × (2 ft. 6 in.) × (3 ft. 4 in.) = $(4\frac{5}{12} \times 2\frac{1}{2} \times 3\frac{1}{3})$ solid feet,

$$= \frac{53}{12} \times \frac{5}{\cancel{2}} \times \frac{\overset{5}{\cancel{10}}}{3} \text{ ft.} = \frac{53 \times 25}{36} \text{ ft.} = \frac{1325}{36} \text{ ft.} = 36\tfrac{29}{36} \text{ solid feet.}$$

If the dimensions of the solid be expressed in feet, inches, and twelfths, the above method of Cross Multiplication may be employed, and we shall find the result expressed in a form similar to the last, but with terms descending lower than solid inches, which was the lowest denomination in the last Example.

184. Since length × breadth × depth = volume (I.), therefore length × breadth = $\frac{\text{volume}}{\text{depth}}$; and since length × breadth is of two dimensions, therefore $\frac{\text{volume}}{\text{depth}}$ is of two dimensions, *i. e.* $\frac{\text{3 dimensions}}{\text{1 dimension}}$ gives two dimensions.

So also, from (I.) length = $\frac{\text{volume}}{\text{breadth} \times \text{depth}}$; *i. e.* $\frac{\text{3 dimensions}}{\text{2 dimensions}}$ gives one dimension: and any fraction in which numerator and denominator are of the same dimension, is an abstract number. (See Art. 67.)

185. The following Exs., though not involving Cross Multiplication, are amongst the most useful of those in which the measurement of surfaces and solids occurs.

Ex. I. Find the number of acres in a rectangular field, of which the length is 35 chains 72 links, and the breadth 24 chains 6 links.

To understand this question, a pupil must know that large pieces of land are measured by means of a chain called *Gunter's Chain*, which is four poles, or 22 yards, in length, and is divided into 100 equal parts, called *Links*.

Also, an acre is equal to a rectangular surface of which the length is 40 poles, or 10 chains, and breadth 4 poles, or 1 chain: hence, the area of an acre = 10 chains × 1 chain = 1000 links × 100 links = 100,000 square links. Consequently, if the dimensions of a field be expressed in links, and its area thence be obtained in square links, this value, when divided by 100,000, will be expressed in acres; *i. e.* if five places be pointed off as a decimal, the result will be acres and decimal parts of an acre, which can be reduced to roods and poles. Returning now to the Ex., we have

$$
\begin{array}{rr}
\text{Length} = 35 \text{ chains } 72 \text{ links} = & 3572 \text{ links} \\
\text{Breadth} = 24 \text{ chains } 8 \text{ links} = & \underline{2408} \text{ links} \\
& 28576 \phantom{\text{ links}} \\
& 142880 \phantom{\text{ links}} \\
& \underline{7144 } \phantom{\text{ links}} \\
\text{Area} = \text{acres} & 86{\cdot}01376 \phantom{\text{ links}} \\
& \underline{4} \phantom{\text{ links}} \\
\text{Roods} & {\cdot}05504 \phantom{\text{ links}} \\
& \underline{40} \phantom{\text{ links}} \\
\text{Poles} & \underline{\underline{2{\cdot}20160}} \phantom{\text{ links}}
\end{array}
$$

and therefore the field contains 86 a. 0 r. 2¼ poles, nearly.

Ex. II. Find how many gallons are contained in a cistern of which the length is 40 inches, breadth 36 inches, and depth 16 inches.

It must here be observed that an imperial gallon is equal to 277·274 cubic inches: hence, when the number of solid inches in any volume is known, the number of gallons which it will contain is found by dividing those solid inches by 277·274. In the above Ex. we therefore have

$$\text{the solid content} = (40 \times 36 \times 16) \text{ solid inches};$$

$$\text{and therefore number of gallons contained} = \frac{23040}{277{\cdot}274} = 83, \text{ nearly.}$$

Ex. III. A roof of 27½ feet by 18¾ is to be covered with lead weighing 8 lbs. per square foot: what would it cost at the rate of £5 4s. for 5 cwt.?

$$
\begin{aligned}
\text{The area of the roof} &= 27\tfrac{1}{2} \text{ ft.} \times 18\tfrac{3}{4} \text{ ft.} \\
&= \left(\frac{55}{2} \times \frac{75}{4}\right) \text{ sq. ft.} \\
&= \frac{4125}{8} \text{ sq. ft.} \\
\text{hence, weight of lead} &= \frac{4125}{8} \times 8 \text{ lbs.} = 4125 \text{ lbs.}
\end{aligned}
$$

To find the cost of the lead, we have the following statement:

$$5 \times 112 \text{ lbs.} \;:\; 4125 \text{ lbs.} \;::\; \pounds 5\ 4\text{s.}$$

The fourth term will be found to be £38 6s. 0¾d.

Exs. 61. Form the following products:—

1. 3 ft. 2 in. × 4 ft. 9 in. × 5 ft. 7 in.
2. 15 sq. ft. 73 sq. in. × 2 ft. 6 in.
3. 11 sq. ft. 9 sup. pr. 3 sec. × 9 ft. 7 in.
4. 2 ft. 4 in. × 3 in. × 11 in.

5. Find the capacity of a rectangular cistern, 12 ft. 3 in. long, 5 ft. 7 in. broad, and 2 ft. 11 in. deep.

6. How many bricks, of which the length, breadth, and thickness are 12, 9, 6 inches respectively, will be required to build a wall, whereof the length, height, and thickness are 64, 9, and $1\frac{1}{2}$ feet?

7. What is the price of a block of stone, of which the length, breadth, and thickness are 37 ft. 8 in.; 8 ft.; and 6 ft. 5 in., at 5s. 6d. per sol. foot?

8. How many square feet of board would be required to make a rectangular box, of which the length, breadth, and depth are respectively $3\frac{1}{2}$ ft., $2\frac{1}{4}$ ft., and 1 ft. $2\frac{7}{8}$ in. respectively?

9. A cubic inch of water weighs 252·458 grains, required the weight (in lbs. Troy) of water in a full cistern $10\frac{1}{2}$ ft. long, $5\frac{1}{4}$ ft. wide, and 11 in. deep.

10. The bottom of a cistern is rectangular, and contains 15 sq. feet, 58 sq. in.; how deep must it be to hold 164 gallons, if a gallon contain $277\frac{1}{4}$ cubic inches?

11. The length of a rectangular field is 25 chains 37 links, and the breadth 17 chains 35 links; find the number of acres contained.

12. Find the number of chains and links in the breadth of a field, whereof the length is 35 chains 15 links, and the area is 45 a. 2 r. $31\frac{1}{2}$ p.

EXTRACTION OF ROOTS.

186. We have already seen (92) that a number, by being successively multiplied by itself, is said to be raised to a power; and the order of the power, whether it be second, third, fourth, &c., depends upon the number of times the original number is to be repeated. This process is termed *Involution;* and the reverse process of obtaining the original number from the power is called *Evolution.* The original number is called, with respect to its power, the *Root;* and this evolution is also termed *Extraction of Roots.*

Thus, since 5×5, or 5^2, or 5 squared (176), as it is called, $= 25$; 5 is called the Square Root of 25. Similarly, since $6 \times 6 \times 6$, or 6^3, or 6 cubed $(181) = 216$, then 216 is termed the cube or third power of 6, and 6 the cube root or third root of 216.

The sign $\sqrt{}$ is used to express the operation of extracting a root; and a small figure, placed thus $\sqrt[3]{}$, shows what root [here the *third* root] is to be extracted: but the figure is generally omitted when the sign refers to the square root; and the sign $\sqrt{}$ itself indicates extraction of the Square Root.

187. Any number or quantity which is thus formed by the squaring, cubing, &c. of any number, is called a complete or *perfect* square, cube, &c.; and therefore the square, cube, &c. root, of such a power can be exactly extracted: but any number, as 20, which lies between two complete squares, 16 and 25, cannot have its root obtained exactly. But since 16 has a root 4, and 25 has a root 5, therefore the root of 20 will be between 4 and 5, or its value will be approximately expressed by the mixed decimal 4·472..; and similarly of every number which lies between two complete squares. So also every number which lies between two complete cubes must have its cube root expressed in a decimal form.

188. If a number be raised to a power, and then the same root of that power be extracted, we shall obtain the original quantity; therefore $\sqrt{3^2} = 3$; $\sqrt[3]{5^3} = 5$.

Also, since the square root of a given number may be defined to be that number which, when multiplied by itself, will produce the given number, therefore $\sqrt{2} \times \sqrt{2} = 2$: so also, $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2} = 2$. Such quantities as $\sqrt{2}$, $\sqrt[3]{2}$, are called *Surds*, or *Irrational* quantities.

Obs. Since fourth, fifth, &c. powers of numbers can be obtained, of course fourth, fifth, &c. roots must also exist; but we shall here confine ourselves principally to the finding of square and cube roots.

What has been said of the raising of whole numbers to powers is also true of fractional quantities, whether vulgar, or commensurable decimals. Thus since $\frac{2}{3}\times\frac{2}{3}$, or $\left(\frac{2}{3}\right)^2=\frac{4}{9}$; therefore $\sqrt{\frac{4}{9}}=\frac{2}{3}$ which equals $\frac{\sqrt{4}}{\sqrt{9}}$; *i. e.* the square root of a fr^n must be obtained by taking the square root of numerator and denominator. In like manner, since $\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$, or $\left(\frac{2}{3}\right)^3=\frac{8}{27}$; therefore $\sqrt[3]{\frac{8}{27}}=\frac{2}{3}$ which equals $\frac{\sqrt[3]{8}}{\sqrt[3]{27}}$; *i. e.* the cube root of a fraction will be found by taking the cube root of both numerator and denominator. If the root of a mixed number be required, it must be reduced to an improper fraction or a decimal. Thus:

$$\sqrt{1\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{\sqrt{25}}{\sqrt{16}}=\frac{5}{4}=1\tfrac{1}{4}.$$

We shall presently show how to extract the square root of decimals as well as of whole numbers.

SQUARE ROOT.

189. Before proceeding to investigate a rule for the extraction of a Square Root, we must remember the following:

Digits,	1,	2,	3,	4,	5,	6,	7,	8,	9.
Squares,	1,	4,	9,	16,	25,	36,	49,	64,	81.

From what has been said, it is plain that the square of 1, is 1; of 10, is 100; of 100, is 10,000, &c. &c.

Therefore the square root of		1	is	1
"	"	100	"	10
"	"	10,000	"	100
"	"	1,000,000	"	1000 &c. &c.

Hence, if a number lie between 1 and 100, *i. e.* have not more than two figures, its square root is between 1 and 10, or has not more than one figure.

If between 100 and 10,000, *i. e.* have not more than four figures, the root is between 10 and 100, or has not more than two figures.

If between 10,000 and 1,000,000, *i. e.* have not more than six figures, the square root is between 100 and 1,000, or has not more than three figures: and so on.

So that if over every alternate figure of any number, beginning at the units' place, a point be placed, the number of points will show the number of figures in the square root. For example, 3757 has more than two figures, and not more than four, or lies between 100 and 10,000, and therefore its root lies between 10 and 100, *i. e.* has not more than two figures, and the number must therefore be thus pointed, $3\dot{7}5\dot{7}$; so also $\dot{1}3\dot{4}5\dot{9}$ is correctly pointed, for there will be three figures in the square root. The divisions which are formed by these points are called *periods*. The periods in $\dot{1}3\dot{4}5\dot{9}$ are 1, 34, and 59.

Also, since the square of ·1 is ·01, therefore

the square root of ·01 is ·1
So also, of ·0001 is ·01
" " ·000,001 is ·001 &c. &c.

It hence appears that in the square of any decimal an even number of decimal places will always be found; and that if there be an odd number in any proposed Ex., the number of places must be made even, by appending a cipher, which cannot alter the value of the decimal. We then point every alternate figure of a decimal, from left to right, beginning with the second figure from the units' place, so that the last figure will always be pointed, in decimals as well as in whole numbers.

190. The Rule for the Extraction of a Square Root is derived from an algebraical operation, wherein a complete square is taken, and a process is then contrived by which the root, which is already known, can be deduced from the complete square.

I shall therefore proceed, contrary to my usual method, to give a Rule for extracting the Square Root, without any more explanation than is necessary for the mere working of Exs.: and the arithmetical illustration of the above algebraical process, which I shall afterwards give, may be *read* or omitted at the reader's pleasure.

RULE. Divide the given number into periods. Find the greatest square number which is not greater than the first period; subtract it from that period, and place the root of the number in the quotient.

To the remainder, after subtraction, bring down two figures or one period, as in Long Division, and consider the whole as a dividend.

For a divisor, double the quotient, and try how often it is contained in the dividend, except the last figure: the figure thus obtained by division place in the quotient, and annex it to the divisor.

Multiply this whole divisor by the last figure in the quotient, and subtract this *subtrahend* from the dividend.

Bring down another period; find a fresh divisor by adding the last figure of the former divisor to that divisor, and proceed exactly as before.

OBS. It will be found that on dividing by the *incomplete* divisor—especially when the early figures in the quotient are small, and the latter ones large—there will result a quotient larger than the one which must be taken. The reason of this will be explained hereafter.

I will work one Ex. which gives no remainder, *i. e.* where the given number is a perfect square, viz. 81796.

```
      8̇17̇96̇ (286
      4
 48)  417
      384
566)  3396
      3396
```

The greatest square number not greater than the first period, 8, is 4; I therefore subtract 4 from the 8, and place in the quotient, 2, the root of the 4: to the remainder, 4, I bring down the second period, 17, and consider 417 as my dividend. For a divisor, I double the 2 in the quotient: and on dividing 41 by this divisor 4, I obtain a quotient 10; but upon trial I find that 8 is the largest quotient that can be employed. I place the 8 in the quotient, and at the right of the 4 in the divisor, making a *complete* divisor 48: multiplying this 48 by the 8, I have a subtrahend 384; the remainder, after subtraction, is 33, and with the third period 96 gives a new dividend 3396: adding the last figure 8 in the divisor 48, to that 48,

I have, as new partial divisor, 56, which goes six times in 339: placing the 6 in the quotient and in the divisor, and multiplying the whole divisor 566 by the 6, I have a subtrahend 3396, which, upon subtraction, leaves no remainder. The square root is therefore 286.

Since most of the numbers we meet with are not perfect squares, we can obtain only approximate roots of such numbers; and the operation is carried to about three places of decimals in the root, requiring of course, six places in the number. If the given number be an integer, or have not so many as six decimal places, the number must be made up by appending ciphers.

Ex. II. Find the square root of 876·535.

$$\begin{array}{rl}
 & \dot{8}7\dot{6}\cdot5\dot{3}5\dot{0}0\dot{0}\ (29\cdot606.. \\
 & 4 \\
49) & 476 \\
 & 441 \\
586) & 3553 \\
 & 3516 \\
59206) & 375000 \\
 & 355236 \\
 & 19764
\end{array}$$

I append three ciphers, and point according to (189). When the third divisor, 592, is obtained, the quotient is 0; I therefore place the 0, as usual, in the quotient and in the divisor, then bring down another period, and proceed as before. Since there were two periods in the integral part of the given number, there will be two integers in the root, and the decimal point must be placed after the 29.

191. I will now illustrate the algebraical process mentioned in (190), as far as can be done in arithmetic.

Let the square number 169 be taken, the root of which is 13. If this number and its root be expressed in the required algebraic form, they will be respectively written 100 + 60 + 9, and 10 + 3.

$$\begin{array}{rl}
 & 100+60+9\ (10+3 \\
 & 100 \\
20+3) & 60+9 \\
 & 60+9
\end{array}$$

Placing in the quotient the 10 + 3, which is known to be the root, I observe that the first part of the root, viz. 10, is the square root of 100, the first part of the number: I may therefore consider that the square root of the 100 gives the 10, and the remainder 60 + 9 is to furnish the 3. Observing the former part of the remainder, viz. 60, I notice, that if it were divided by twice the 10 in the quotient, I should obtain the required number, 3: I therefore make 20, *i. e.* twice the quotient, my divisor. Also, since there must be no remainder, I must have as subtrahend 60 + 9; and since the 9 is the square of 3, therefore, if I append the 3 to the 20 in the

divisor by the sign (+), and multiply the whole divisor $20+3$ by 3, I shall obtain a subtrahend equal to the dividend $60+9$, and therefore shall have no remainder, as was required.

An algebraical process similar to the above would *prove* that this method of procuring a divisor and subtrahend would *always* succeed in bringing no remainder, *i. e.* in obtaining the exact root of a complete square. But this Ex. of course only *illustrates* the method of proof, and shows how the Rule is algebraically deduced. The operation, when condensed into an arithmetical form, stands thus:

$$\begin{array}{rl} & \dot{1}6\dot{9}\ (13 \\ & 1 \\ 23) & \overline{69} \\ & \underline{69} \end{array}$$

192. I will work one more Ex. which will show how an error in the quotient may arise in dividing by an incomplete divisor.

$(27)^2 = 729$, which, algebraically represented, is $400+280+49$: and the quotient or root, is $20+7$.

And this, when condensed is

$$\begin{array}{rl} & 400+280+49\ (20+7 \\ & 400 \\ 40+7) & \overline{\quad 280+49} \\ & \underline{\quad 280+49} \end{array} \qquad \begin{array}{rl} & \dot{7}2\dot{9}\ (27 \\ & 4 \\ 47) & \overline{329} \\ & \underline{329} \end{array}$$

Obs. Since to divide 280 or 320 by 40 is the same as to divide 28 or 32 by 4; I may therefore call 4 the divisor, when 28 or 32 is the dividend, and consider 40 as the divisor, when 280 or 320 is the dividend.

If this Ex. be worked in the usual manner, it will be found that the incomplete divisor 4 will go eight times in 32, whereas the real quotient is found to be only 7. By observing the algebraic method, I learn, that when the dividend 329 is separated into its parts $280 + 49$, the true quotient is obtained by dividing only the former part, 280,

by the 40, or 28 by 4 : but in the arithmetical operation I cannot see how much of 329 is the former part which will give me a correct quotient; and therefore I have to divide the whole 329 by 40, or 32 by 4, and so run the risk of an error. If the second part of the dividend 49 had been less than 40, the quotient would at once have appeared to be 7, and no error arisen. Hence, the larger the second part of divisor (as 40), the greater will be the error: and since the the dividend (as 49) is, when compared with the incomplete 4 in the divisor was obtained from the *former* part of the quotient, viz. 2, and the 49 from the latter part, viz. 7, the error will be the greatest when the earlier figures of the quotient are small, and the latter are large.

This method of proof is not limited to numbers, the roots of which consist but of two figures: but any attempt to extend the illustration would be very cumbrous without the use of algebra.

Exs. 62.

Extract the Square Root of each of the following numbers:—

1. 729	4. 2832489	7. $1241\frac{35}{169}$	10. 197·4025
2. 11025	5. $27\frac{9}{16}$	8. $12122\frac{1}{100}$	11. 36·343
3. 8264446281	6. $371\frac{46}{49}$	9. 10·5625	12. ·002401..

193. We have seen that the area of a square is obtained by squaring any one of its sides; hence, the number in the side of a square is found by extracting the square root of the number which represents the area of the square.

The number which represents the area of a square figure may not always be a complete square: for instance, if in Fig. 1, p. 171, a square be described with sides equal to *ac* or $\sqrt{2}$, then its area would be $(\sqrt{2})^2 = 2$, which is not a perfect square. When, therefore, the number representing *the area* of a square is not itself a perfect square, the

number representing the side will be incommensurable or irrational.

Quantities which are not perfect squares, cubes, &c. may be made so by multiplying them by certain factors. For, in order that a number may be a complete square, each of its factors must be contained 2, 4, 6, &c. times, *i. e.* some multiple of *two* times; to be a complete cube, each must be contained 3, 6, 9, &c. times, *i. e.* some multiple of *three* times; and so on for higher powers. For example, the number 144 will be found $= 2^4 \times 3^2$, where the indices 4, and 2, are multiples of 2, and therefore 144 is a perfect *square*. So also $1728 = 2^6 \times 3^3$, where the indices are multiples of 3, and therefore 1728 is a perfect *cube*.

Hence, if I resolve any number into its factors, I can tell by inspection whether it be a perfect square, cube, &c.; and if not, what additional factors must be introduced into it to make it so. Thus, $20 = 2^2 \times 5$; and since the index of the 5 is not a multiple of 2, it must be made so, by introducing an additional 5; it then becomes $2^2 \times 5^2$, or 100, a perfect square. Again, to make 48 a perfect cube, I observe that $48 = 2^4 \times 3$; and in order that the indices may be multiples of 3, this must be changed into $2^6 \times 3^3$, and therefore be multiplied by $2^2 \times 3^2$, *i. e.* by 36; and we then shall have $48 \times 36 = 1728$ a perfect cube.

In obtaining the square root of a fractional quantity, we may extract the root of numerator and denominator, if the denominator be a complete square, as in (188);

$$\text{Thus, } \sqrt{\frac{729}{1235}} = \frac{27}{35}$$

or taking the case of a mixed number,

$$\sqrt{4\frac{49}{144}} = \sqrt{\frac{625}{144}} = \frac{25}{12} = 2\tfrac{1}{12}$$

But if the den^r be not a perfect square, it is better to reduce the fraction to a decimal, and then extract the root. For example, if I have to extract the square root of $27\frac{27}{128}$, then since $27\frac{27}{128} = 27{\cdot}2109375$, a single extraction will suffice to give the answer at once; whereas, trying the former method, I should have

$$\sqrt{27\tfrac{27}{128}} = \sqrt{\frac{3483}{128}} = \frac{59{\cdot}01\ldots\ldots}{11{\cdot}31\ldots\ldots},$$

and to obtain the result, I must extract *two* roots, and perform an operation in Long Division.

Again, we may make the den^r a complete square, as described in the preceding paragraph; and then, since the root of the den^r will be known by the very process of completing the square, it will be necessary to extract only the root of the num^r. Thus, taking the above example, and observing that a factor 2 will make the 128 become 256... or 14^2, we have

$$\sqrt{27\tfrac{27}{128}} = \sqrt{\frac{3483}{128}} = \sqrt{\frac{6966}{256}} = \frac{83{\cdot}462\ldots}{16} = 5{\cdot}216\ldots.$$

In finding the root of a circulating decimal, it will sometimes happen that the equivalent vulgar fraction will have both numerator and denominator complete squares, and we can then readily extract the root. Thus:

$$\sqrt{1{\cdot}\dot{7}} = \sqrt{1\tfrac{7}{9}} = \sqrt{\tfrac{16}{9}} = \tfrac{4}{3} = 1\tfrac{1}{3}, \text{ or } 1{\cdot}\dot{3}.$$

But as this is rarely the case, we must generally find the approximate root by the usual method, though it will not be a recurring decimal.

194. Sometimes a fraction involving a surd may be in its lowest terms, but yet not in a form the most convenient for finding its value. Thus $\frac{1}{\sqrt{2}}$ is in the lowest terms; but

multiplying numerator and denominator by $\sqrt{2}$, it becomes $\frac{\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{\sqrt{2}}{2}$, or $\frac{1}{2}\sqrt{2}$. And we shall find that this fraction is more simple than the former one. For, since $\sqrt{2} = 1{\cdot}4142...$ therefore $\frac{1}{\sqrt{2}} = \frac{1}{1{\cdot}4142...}$ which will require a Long Division sum to bring out its value; but $\frac{1}{2}\sqrt{2} = \frac{1}{2}(1{\cdot}4142...) = {\cdot}7071$ ...... by mere Short Division. Hence $\frac{1}{2}\sqrt{2}$ is more simple than $\frac{1}{\sqrt{2}}$.

And as a general rule, all quantities involving surds are in their simplest form, when the surds are in the numerator and not in the denominator.

The following Exs. illustrate the operations of the last few pages.

Ex. I. A square field contains 15 a. 2 r. 20 p. Find its side in chains.

15 a. 2 r. 20 p. = 2500 sq. poles; therefore the side of a field containing 2500 sq. poles = 50 linear poles

$$= \frac{50}{4} \text{ linear chains (since 4 poles = 1 chain.)}$$
$$= 12\tfrac{1}{2} \text{ linear chains.}$$

Ex. II. Two acres of land are to be cut from a rectangular field of which the breadth is 20 chains 50 links, by a line parallel to it. Find the length of the plot.

1 acre = 40 p. × 4 poles = 10 chains × 1 chain = 10 sq. chains; and length × breadth = 2 acres; or, since the breadth is $2\frac{1}{2}$ chains, therefore length × $2\frac{1}{2}$ chains = 2 acres = 2 × 10 sq. chains = 20 sq. chains.

$$\text{Hence length} = \frac{20 \text{ sq. chains}}{2\frac{1}{2} \text{ linear chains}}$$
$$= \overset{4}{\cancel{20}} \times \frac{2}{\cancel{5}} \text{ linear chains (184)}$$
$$= 8 \text{ linear chains.}$$

Exs. 63. Find the value (to 4 places of decimals) of

1. $\sqrt{3}$ 2. $\frac{1}{\sqrt{3}}$ 3. $\sqrt{6{\cdot}24\dot{9}}$ 4. $\sqrt{15{\cdot}\dot{3}}$

5. Find the side of a square field, whose area is equal to that of a rectangle 1800 yds. by 800 yds.

6. The si 'es of 2 squares are 15 ft. and 25 ft.; find the side of another which shall equal the sum of the two former.

7. A rectangular field measures 64 ft. in length by 48 ft. in breadth; what is the length of the diagonal? *

8. A square field contains 3 a. 3 r. $2\frac{53}{121}$ p.; find the length of its side in yards.

9. The painting of a square area, at $7\frac{1}{2}$d. per square yard, comes to £22 15s. $7\frac{1}{2}$d.; find the length of the side of the square.

10. Two sides of a triangle are respectively 236·25 and 243·75 ft., also the altitude is $2\frac{2}{3}$ ft.; find the length of the base.*

CUBE ROOT.

195. The following numbers must be first remembered:

Digits,	1,	2,	3,	4,	5,	6,	7,	8,	9.
Cubes,	1,	8,	27,	64,	125,	216,	343,	512,	729:

and it will be observed that no two of these cubes end with the same digit: hence, if a given number be a perfect cube, the last figure in its root may be known. For instance, if a cube number end in 3, its cube root ends in 7: if the number end in 2, its root ends in 8. Recollection of this fact is often of service.

According to the method of (181), we learn that the cube root

of	1	is	1
„	1,000	„	10
„	1,000,000	„	100 &c. &c.

hence, if a number lie between 1 and 1000, *i. e.* have not more than three figures, its cube root is between 1 and 10, or has not more than one figure.

If the number be between 1000 and 1,000,000, *i. e.* have not more than six figures, its root has not more than two figures: and if the number have not more than nine figures, its root has not more than three figures.

So that if over every third figure, beginning at the units' place, a point be placed, the number of points will show the number of figures in the cube root.

Similarly, since the cube of ·1	is	·001
therefore the cube root of ·001	„	·1
So also,................ ·000,001	„	·01
„ „ „ ·000,000,001	„	·001 &c. &c.

* See Appendix, Note to *Art. Ratio*.

It hence appears that the number of places in the cube of any decimal must always be some multiple of 3: and if the number of places in any decimal, of which we have to find the cube root, be not a multiple of 3, it must be made so by appending ciphers: we therefore commence pointing at the units' place, and point every third figure to the left over the integers; and to the right over the decimals, if there be any.

I shall now give two Rules for the extraction of the Cube Root, since each has its merits. The former is comparatively short, and can be rendered intelligible to any one possessing a slight knowledge of Algebra; and the proof of the Rule is therefore given in a note. But in Exs. where the Root contains several figures, the work becomes exceedingly heavy, and the Second Rule is then preferable. This Second Rule, called Horner's method, has the merit of exhibiting all the work in a very convenient form, especially when the given number is large. I, however, consider it very difficult for a learner to remember; and the proof of it would be out of place here, as it involves some knowledge of the Theory of Equations.

FIRST RULE. Divide the given number into periods. Find the greatest cube number which is not greater than the first period. Subtract it from that period; place the root of the number in the quotient, and form a dividend, as in Square Root. For a divisor, multiply the square of the quotient by 3, and try how often it is contained in the dividend, except the last two figures; the figure thus obtained by division place in the quotient.

To form the subtrahend: consider the quotient made up of two parts, whereof one is the last figure alone, and the other is the former figure or figures with a cipher annexed. Cube this last figure, and to it add three times the product of the two parts of the quotient by the whole quotient.

Subtract this subtrahend from the dividend; to the

remainder, if any, bring down another period,—find a fresh divisor by again multiplying the square of the quotient by 3, and proceed precisely as before.*

OBS. As in Square Root, the divisor often gives upon trial too large a quotient.

Ex. Find the cube root of 185193.

$$\begin{array}{rl} & 18\dot{5}19\dot{3}\ (57 \\ & \underline{125} \\ 3\times 5^2 = 75) & \overline{60193} \\ & \overline{343} = 7^3 \\ & \underline{59850} = 3\times 50\times 7\times 57 \\ & \underline{\underline{60193}} \end{array}$$

Placing a point over the 3 and the 5, I find that 5^3, or 125, is the greatest cube number below 185. I therefore subtract 125 from 185, place 5 in the quotient, and bringing down the next period, have 60193 as dividend. The divisor $= 3 \times 5^2 = 75$, which appears to go eight times, but the real quotient is 7. According to the Rule, I now consider the quotient 57 as composed of two parts, 50 and 7: and for a subtrahend I take first 7^3, or 343: next, I have $3 \times 50 \times 7 \times 57 = 59850$; and the sum of these results forms a subtrahend 60193, which, when subtracted from the dividend, leaves no remainder. Hence 57 is the exact cube root of 185193.

---

* Let $(a+b)^3$, or $a^3 + 3a^2b + 3ab^2 + b^3$ be a perfect cube, from which I am to deduce a rule for the Extraction of the Cube Root. Of course the required root is $a + b$.

$$\begin{array}{rl} & a^3 + 3a^2b + 3ab^2 + b^3 \quad (a+b \\ & \underline{a^3} \\ 3a^2) & \overline{3a^2b + 3ab^2 + b^3} \\ & \qquad\qquad\quad b^3 = b^3 \\ & \underline{3a^2b + 3ab^2} \quad = 3ab(a+b) \end{array}$$

Here it is plain that the first portion of the root is found by taking the cube root of a^3, the first portion of the given quantity. Also b, the second part of the root, is obtained by dividing $3a^2b$, the first portion of the dividend, by $3a^2$, or three times the square of the a already obtained in the root.

The subtrahend is formed of two parts, b^3 and $3a^2b + 3ab^2$: this latter portion $= 3ab(a+b)$: and these two algebraical expressions, when enunciated in words, give the process for forming the subtrahend mentioned in the Rule.

Since in Arithmetical Examples the dividend is expressed as one quantity, but the true quotient b is found by dividing only a *part* of that dividend; therefore if the latter portion of the dividend, viz. $3ab^2 + b^3$, be large in proportion to the former part, $3a^2b$, then the quotient obtained by division will be too large; and the error will be greatest when b is large and a small. If there be more than two terms in the quotient, b must always represent the last term alone, and a the whole of the other terms.

Also, since in division in Algebra, the first term of the divisor is alone useful in finding the quotient, therefore (in the above method of working Algebraical Examples,) since $3a^2$ will always be the first term of the divisor, however many terms may be in the quotient, the first divisor, $3a^2$, will serve for every dividend throughout the *Example*.

196. The following Ex. involves decimals, and gives only an approximate root.

$$
\begin{array}{rl}
 & 2\dot{1}03\dot{5}\cdot80\dot{6}\ (27\cdot6 \\
 & 8 \\
3 \times 2^2 = 12) & \overline{13035} \\
 & 343 = 7^3 \\
 & 11340 = 3 \times 20 \times 7 \times 27 \\
 & \overline{11683} \\
3 \times (27)^2 = 2187) & \overline{1352800} \\
 & 216 = 6^3 \\
 & 1341360 = 3 \times 270 \times 6 \times 276 \\
 & \overline{1341576} \\
 & \overline{11224}
\end{array}
$$

I here observe that in forming the second subtrahend, I consider the quotient as made up of 270 and 6. By bringing down more periods of three ciphers, I might carry the root to several places of decimals; but the work becomes exceedingly heavy when the quotient extends beyond three figures. Since one period consisted of decimals, the last figure in the quotient must be pointed off as a decimal.

Second Rule. I. Divide the given number into periods. Find the greatest cube number which is not greater than the first period. Subtract it from that period; place the root of the number in the quotient, and form a dividend as in Square Root.

II. To the left of the number, and at some distance from it, place two columns (A) and (B). Under (A) insert three times the root, and under (B) three times its square. Annex one cipher to (A), and two ciphers to (B), and with (B) as divisor, find the next figure in the quotient.

III. Add this figure to (A); also add to (B) the product of (A) by the last figure in (A); and multiply the sum by the last figure in the root, to form a subtrahend, which subtract from the dividend and bring down one period.

IV. Under (A) place twice its units' figure, and under (B) the square of that figure. Find the sum of the last

two lines in (A), and of the last *three* in (B), and again annex one cipher to (A) and two to (B). With (B) as divisor, find another figure in the root, and proceed to form another subtrahend, as in (III.) of this Rule.

The following Example will be found to exemplify this Rule; and the lines have been written widely to indicate the successive steps from (A) to (B), and from (B) to the subtrahends.

A		B		$2\dot{1}71\dot{7}63\dot{9}$ (279
60				8
7		1200 = total divisor)		13717
67 × 7	=	469		
14		1669 × 7	=	11683
810		49		
9		218700		) 2034639
819 × 9	=	7371		
		226071 × 9	=	2034639
				....

Since the first figure in the root is 2, I place 3 × 2, or 6, under (A), and 3×2^2, or 12, under (B), annex one cipher to (A) and two ciphers to (B). Using this 1200 as a divisor, I have 7 as quotient, which I add to (A), making 67, and to (B) I add 469, the product of this 67 by 7, making 1669: I multiply this 1669 by the same 7, making a subtrahend 11683, and obtain a dividend 2034639. I now place under (A) twice its last figure 7, and under (B) I place 7^2 or 49; I now add the last *two* lines in (A), making 81, and the last *three* lines in (B), viz. 49, 1669, and 469, obtaining 2187, then annex one cipher to (A), two ciphers to (B), and making this 218700 in (B) as a divisor, I obtain a figure 9 in the root. I now add this 9 to the 810 in (A), and multiply the whole of it, or 819, by 9, making 7371, then add it to (B), making 226071; lastly multiply this by the same 9, placing the product as a subtrahend. If there had been any more periods, I should have commenced forming a fresh divisor as in (IV.)

197. The observations which were made upon the extraction of the Square Root of Fractions in (193) apply also to a similar use of the Cube Root. Also, the probability of error in obtaining the figures in the quotient from the use of the divisor may be noticed from the reasoning in (192).

Just as the number in the side of an area of square form is found by extracting the square root of the number representing the area: so the number in the edge of a volume of cubic form is found by extracting the cube root of the number which represents the volume.

I will work one or two Exs. to illustrate the application of Cube Root.

Ex. IV. A beam is 6 ft. 9 in. long, 2 ft. 3 in. broad, and 9 in. thick; required the side of a cube of equal capacity.

Working fractionally we have

Solid content of beam $= 6\frac{3}{4}$ ft. $\times 2\frac{1}{4}$ ft. $\times \frac{3}{4}$ ft.

$$= \left(\frac{27}{4} \times \frac{9}{4} \times \frac{3}{4}\right) \text{ sol. ft.} = \frac{729}{64} \text{ sol. ft.}$$

$$\text{or} = \frac{9^3}{4^3} \text{ sol. feet.}$$

therefore, edge of the cube $= \sqrt[3]{\frac{9^3}{4^3}}$ sol. ft. $= \frac{9}{4}$ lin. feet $= 2\frac{1}{4}$ linear feet.

Ex. V. Find the area of any one of the six surfaces of a cube containing 11 cubic feet, 675 cubic inches.

To find the area of this surface, I must find the edge of the cube, and square it. Reducing the cubic feet to inches, I have

$$\text{volume of cube} = 19683 \text{ cubic inches;}$$

therefore, edge of the cube $= \sqrt[3]{19683} = 27$ linear inches $= 2$ ft. 5 in.;

and therefore area of the side $= (2\frac{1}{4})^2 = \left(\frac{9}{4}\right)^2 = \frac{81}{16}$ sq. ft.

$= 5\frac{1}{16}$ sq. ft.

$= 5$ sq. ft. 0 pr. 9 sq. in.

Ex. VI. If the volume of any cylinder equals its length × area of its base, find the value of this area in a cylindrical wire 50 feet long, and made out of a square inch plate of metal ·05 inches in thickness.

Here, since area × length = whole volume of metal

= 1 sq. inch × thickness of the plate;

or, since Area × 50 ft. = 1 sq. inch × ·05 inches;

$$\text{therefore Area} = \frac{1 \text{ sq. in} \times \cdot 05 \text{ in.}}{50 \times 12 \text{ in.}} \qquad \text{(I)}$$

$$= \frac{\cdot 05}{50 \times 12} \text{ sq. inches}$$

$$\left(\begin{array}{c}\text{Multiplying numerator} \\ \text{and denominator by } 100\end{array}\right) = \frac{5}{500 \times 12} \text{ sq. inches}$$

$$= \frac{1}{12000} \text{ sq. inches}$$

The right-hand side of (G) gives $\frac{3 \text{ dimensions}}{1 \text{ dimension}}$; which, according to (184), is 2 dimensions, as an Area ought to be.

Obs. An expression which is of the same dimensions throughout, as (I), is said to be *homogeneous*.

Exs. 64. Extract the Cube Root of each of the following numbers:—

1. 42875	5. $42\frac{7}{8}$	9. 77·854483
2. 970299	6. $1367\frac{631}{1000}$	10. 284890·312
3. 6539203	7. $2345\frac{22}{343}$	11. 1334·633301
4. 32798729601	8. $423987\frac{457}{512}$	12. $\frac{\cdot 000405224}{\cdot 064}$

Find the value (to 2 places of decimals) of

13. $\sqrt[3]{15}$ 14. $\sqrt[3]{155}$ 15. $\sqrt[3]{342 \cdot 9}$ 16. $\sqrt[3]{\frac{1}{15}}$

17. Find the edge of a cubical box containing 13824 solid inches.

18. A cistern is 72 ft. long, 24 ft. broad, and 27 ft. deep: find the edge of a cubical cistern of the same content.

19. Find the length of the edge of a cube which contains 94 yds. 14 ft. 1088 inches.

20. Find the whole surface of a cube which contains 15 solid feet and 1080 solid inches.

TO EXTRACT ANY ROOT WHATEVER.

198. Upon the same principle as the Second Rule given above, is the following Rule for the extraction of any root whatever.

I. Divide the given number into periods, each containing *as many* figures as the index of the root to be extracted.

Find the root of the first period, and place it in the quotient.

II. Form, at equal distances from each other, columns, called A, B, C, &c., equal in number to the above index; and consider the first period in the given number as the head of the last column. Under that to the left, place the figure in the root; under the next, the square of that figure; under the next, the cube, and so on. The highest power of this figure will fall under the first period; subtract it from that period, and form a new dividend as usual.

III. To find a trial divisor:—Under (A) again place the figure in the root, multiply (mentally) the sum by this root; but place the product under (B), not under (A); add the two lines in (B), multiply the sum by the root, and place the product under (C); proceed in like manner to the *last column* which stands before the proposed number.

Commence at (A) precisely the same process, and continue it to the *last column but one;* again continue the same process, dropping a column each time, till only the first one is employed. Now annex *one* cipher to (A), *two* to (B), *three* to (C), and so on; and make the result of the column, which precedes the proposed number, a trial divisor of the dividend, thus obtaining another figure in the root. The subtrahend will be obtained presently.

IV. The next trial divisor is obtained by forming a series of products with the new figure, of precisely the same kind as those obtained from the former figure. Also, in forming the first row of these new products, the last one to the right will be the subtrahend from the lately formed dividend, and will after subtraction furnish a new dividend as usual.

Ex. To extract the fifth root of 60466176.

A	B	C	D	E
				60466176 (36
3	9	27	81	243
3	18	81	324	
6	27	108	4050000	) 36166176
3	27	162	1977696	36166176
9	54	270000	6027696	....
3	36	59616		
12	9000	329616		
3	936			
150	9936			
6				
156				

In the above Ex. it may be seen that the first figure in the root, viz. 3, must be found by trial, as in Cube Root. Under the five columns, A, B, C, D, E, there have been placed, 3, 3^2 or 9, 3^3 or 27, 3^4 or 81, 3^5 or 243, this last power being the subtrahend from the first period. Also, this figure 3 has been added to (A) four times; after three of these additions each successive amount has been multiplied by this figure, and the products added to (B). In like manner, of the three amounts thus formed in (B), the first two have been multiplied by the root figure, and the products carried to (C). The amount of the first addition alone in (C) has been multiplied and carried to (D). The results of the columns are now 15, 90, 270, 405, and with the ciphers added become 150, 9000, 270000, 4050000, which last, taken as a trial divisor, gives 6 as the next figure in the root.

Again, after finding the second figure 6 in the root, the subtrahend consists of the last product of the first row that would be formed, to find another trial divisor for a third period.

Exs. 65. Q.

1. Explain the process of finding the area of an oblong, the sides whereof contain any number of feet and inches.

2. Explain by figures the nature of the products, when feet are multiplied by feet, feet by inches, inches by inches, &c. down to twelfths of an inch.

3. Find how much money must be paid for £10800 stock at 84⅞, and 2s. 6d. per cent. brokerage.

4. A mixture is made of 40 lbs. at 3s. 6d., 44 lbs. at 3s. 10½d., and 55 lbs. at 4s. 6d.; what will be the gain per cent. if the mixture be sold *for £39 4s.?*

5. What is the interest of money invested in the $3\frac{1}{2}$ per cents. at $89\frac{3}{8}$?

6. An inclined plane 3 miles long has a total rise of 212·15 feet; find the rise per yard in decimal parts of an inch.

7. Reduce to a simple fraction the ratio between the sum and difference of these expressions:—

$$\left(\tfrac{3}{2}+\tfrac{2}{3}\right) \times \left(\tfrac{5}{4}+\tfrac{4}{5}\right) \text{ and } \left(\tfrac{3}{2}-\tfrac{2}{3}\right) \times \left(\tfrac{5}{4}-\tfrac{4}{5}\right)$$

8. Form the following product, (17 ft. 4 in.) × (18 ft. 7 in.), by Duodecimals, and by Fractions.

9. Shew that $\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$; which is the simpler of the two, and why?

10. Find the square roots of 6424·0225, and of $\frac{729}{1296}$.

11. What is the solid content of a box, of which the height, length, and breadth are respectively 3 ft. 2 in., 5 ft. 4 in., and 2 ft. 7 in.?

12. What is the true discount on a 4 months' bill for £151 7s., drawn on January 1st, and discounted on the 19th of February, at $4\frac{1}{2}$ per cent.?

13. An article which sold for 50 guineas caused a loss of $5\frac{1}{2}$ per cent., what should it have fetched, to produce $7\frac{1}{2}$ per cent. profit?

14. If the rate of exchange between Amsterdam and Paris be 21·1 francs for 10 florins, and between London and Amsterdam be 12·15 florins for £1 sterling, what is the rate of exchange between London and Paris?

R.

1. State two numbers between 1 and 100, such that the first has exact square and cube roots, and the second has exact square and fourth roots.

2. If a cubic foot of air weighs $1\frac{1}{4}$ oz., what is the weight of air contained in a room 25 ft. 6 in. long, 16 ft. broad, and 10 ft. 9 in. high?

3. A cubical block contains 24 ft. 1403 in.; find its edge, and the area of its 6 sides.

4. Find the value of $\sqrt{23\frac{4}{9}}$; and of $\sqrt[3]{163\frac{5}{16}}$ to two places of decimals.

5. What is the extent of surface of a covered box, of which the dimensions are 5 ft. 10 in., 3 ft. 6 in., and 7 ft. 2 in.?

6. If the painting of a room 9 ft. 6 in. high, 15 ft. 3 in. long, and 10 ft. broad, come to £5 7s. 6d.; what will be the expense of painting another of which the length, breadth, and height are 18 ft. 2 in., 11 ft. 7 in., and 12 ft. 8 in.?

7. If 500 rupees produce £505 4s. 2d., what is the rate of exchange between England and the East Indies?

8. If by selling goods at £2 5s. I lose 15 per cent., what would be the loss or gain per cent. if the price were £2 18s. 6d.?

9. The sides of two square pieces of ground are 360 yds. and 160 yds.; find the value of the latter, if the former be worth £300.

10. Railway shares which were purchased at a premium of 50 per cent. and sold at a discount of $25\frac{1}{4}$ per cent., produce a loss of £7550; how much money was invested?

11. The length of a rectangular field is 12 chains 35 links, and the breadth is 10 chains 75 links; find the number of acres contained.

12. A square field contains 9 a. 0 r. 4 p.; find the length of its side in chains.

13. If $\frac{1}{3}$ of a debt be due in 3 months, $\frac{1}{4}$ in 4 months, $\frac{1}{5}$ in 6 months, and the remainder in 10 months; what is the equated time of payment?

14. Explain the meaning of the term "Fractional Quotient."

S.

1. A person employs 6 men for 5 days at 8 hours each, and 5 women for 6 days at 10 hours, at the respective wages of 4d., and $2\frac{1}{4}$d. per hour: how much must he pay them?

2. Find the number of men in a side of a square, if when drawn up in rank and file they number 81 by 36.

3. Reduce to the simplest form $\left(\frac{2}{3} \text{ of } \frac{7}{9} \text{ of } 1\frac{7}{11}\right) \div \left(2\frac{1}{5} \text{ of } 3\frac{4}{7}\right)$.

4. The discount on £153 due half a year hence is £3, what is the rate of interest?

5. In what time will £750 amount to £918 15s. at $4\frac{1}{2}$ per cent. Simple Interest?

6. How many years' purchase should be paid for freehold property to produce $5\frac{1}{4}$ per cent?

7. What is the value of a perpetual annuity of £60, reckoning money worth 4 per cent?

8. A testator bequeaths £500 to *A*, £300 to *B*, and £450 to *C*: but his estate only produces £600; find each man's share.

9. A cubic foot of water weighs 1000 oz.; what is the weight of water in a full cistern, the dimensions of which are 7 ft. 6 in., 5 ft. 2 in., and 3 ft.?

10. The mercury in a barometer rises uniformly from 29·15 to 30·73 in 12 days: find the ratio of the daily rise to the average height.

11. A square field has a diagonal 160 yards long; find the area of the field.*

12. If a solid foot of gold weigh 1260 lb. 8 oz., and a solid foot of cork weigh 15 lb. Troy: how much cork will weigh as much as 1 inch of gold?

13. A cubical mass of metal, of which the edge is 3·35 inches in length, is drawn out into a wire, of which the area of a section is ·561125 square inches, find the length of the wire.

14. Shew how to make any number a complete 2nd, 3rd, 4th, &c. *power*. Ex. Find a multiplier which shall make 75 a perfect cube.

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* See Appendix, Note to Art Ratio.

SCALES OF NOTATION.

199. It will often have been observed that each figure in any number has *ten* times the value that it would have had, if it had been one place to the right. The selection of this number *ten* has arisen from its being the number of figures on the hands; and hence has arisen the use of the term *digits* for the Arabic figures, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Hence 10 is called the Radix of the common Scale* of Notation, and the scale itself is called the *denary* scale, from the Latin distributive numeral DENI, *in tens*.

If any other number be chosen as the radix of a scale, the new scale derives its name from the corresponding Latin term.

Thus, if the radix be	2,	the scale is called the	*Binary*.
"	3	"	*Ternary*.
"	4	"	*Quaternary*.
"	5	"	*Quinary*.
"	6	"	*Senary*.
"	7	"	*Septenary*.
"	8	"	*Octenary*.
"	9	"	*Nonary*.
"	10	"	*Denary*.
"	11	"	*Undenary*.
"	12	"	*Duodenary*.

In the two latter scales, since 11 and 12 digits are required, therefore the letters *t* and *e* are used to indicate *ten* and *eleven*.

200. Now it will be easily seen that any number, as 5372, in the common scale can be written in the following algebraical form,

$$5000 + 300 + 70 + 2, \text{ or } 5 \times 10^3 + 3 \times 10^2 + 7 \times 10 + 2. \qquad \text{(II.)}$$

* The term Scale has here the meaning of the word SCALA, a ladder, whereby we, as it were, *ascend* from lower to higher numbers by steps of 10.

And if 8 were the radix of the scale in which this said number 5372 was arranged, the digits of which it is composed would be different; and if a, b, c, d, represented these new digits, which I do not yet know, I should have

$$a \times 8^3 + b \times 8^2 + c \times 8 + d = 5372. \qquad \text{(II.)}$$

Now, in series (I.), we may observe that if we divide by the radix 10, we shall have as quotient $5 \times 10^2 + 3 \times 10 + 7 + \frac{2}{10}$; *i. e.* $\frac{2}{10}$ is the frac[l] part of the quot[t], or as we commonly express it, 2 is the rem[r]; *i. e.* the last digit of a number in scale 10 is found by dividing the entire number by the radix 10. Also, if we continue successively to divide this quot[t] $5 \times 10^2 + 3 \times 10 + 7$ by 10, we shall get as rem[rs] 7, 3, and 5; *i. e.* the successive digits from right to left are obtained by dividing by the radix, until there be no dividend left. So also, if we divide the left-hand side of (II.) by the new radix 8, the quotient is $a \times 8^2 + b \times 8 + c + \frac{d}{8}$, or the remainder d is the last digit, as 2 was, just above. Now the two sides of equation (II.) have of course precisely the same value, and since when I divide the *left*-hand side of (II.) by 8, I obtain the last digit in scale 8, therefore I shall obtain the same digit by dividing the *right*-hand side 5372 by 8.

$$\begin{array}{llll}
8)\underline{5372} & & 8)\underline{a \times 8^3 + b \times 8^2 + c \times 8 + d} & \\
\;8)\underline{671} - \frac{4}{8} & \text{or rem}^r\ 4 & 8)\underline{a \times 8^2 + b \times 8 + c} & + \frac{d}{8} \text{ or rem}^r\ d \\
\;\;8)\underline{83} - \frac{7}{8} & \text{,,}\quad 7 & 8)\underline{a \times 8 + b} & + \frac{c}{8} \quad \text{,,}\quad c \\
\;\;\;8)\underline{10} - \frac{3}{8} & \text{,,}\quad 3 & 8)\underline{a} & + \frac{b}{8} \quad \text{,,}\quad b \\
\;\;\;\;8)\underline{1} - \frac{2}{8} & \text{,,}\quad 2 & & \frac{a}{8} \quad \text{,,}\quad a \\
\;\;\;\;\;\underline{\underline{0}} - \frac{1}{8} & \text{,,}\quad 1 & &
\end{array}$$

Placing the two quantities in equation (II.) in parallel columns, and *dividing* both by 8, I obtain as quotients, $671\frac{4}{8}$, and $a \times 8^2 + b \times 8 + c + \frac{d}{8}$.

Hence the quotients 671, and $a \times 8^2 + b \times 8 + c$ are equal, and the rem^rs 4 and d are equal; *i. e.* the first of the new digits to the right is 4. The second division by 8 gives equal quotients, viz. $83 = a \times 8 + b$, and a rem^r $c = 7$. The next gives quotients 10 and a respectively, and a rem^r $b = 3$. Hence the values of a, b, c, d, are 10, 3, 7, 4; but since in the scale of 8 there can be no single number $a = 10$, I therefore divide the 10 still further, till I obtain *two* more digits, 2 and 1, in place of the 10, and the entire number of digits will be 1, 2, 3, 7, 4, &c.; and it now appears that there will be 5 figures in the number 5372 when expressed in the scale of 8, and the number will be represented by 12374,

$$\text{or } 1 \times 8^4 + 2 \times 8^3 + 3 \times 8^2 + 7 \times 8 + 4 \qquad \text{(III.)}$$

The number in (III.) is also $= 4096 + 1024 + 192 + 56 + 4$

$$= 5372, \text{ as before.}$$

Hence we learn that if we have to transfer a number from the denary scale to any other scale of notation, we have only to divide the proposed number by the new radix, till there is no integral quotient. The successive remainders will be the new digits, the first one filling the units' place.

The process of reconverting from any of these new scales to the denary is shown in (III.), where the number 12374 in the octenary scale is restored to the denary, giving 5372.

The fact that 5372 in the scale of 10 is equal to 12374 in the scale of 8 is thus briefly exhibited;

$$(5372)_{10} = (12374)_8.$$

201. We have now seen how to convert from the denary scale into any other scale, and from any scale into the denary. But if we wish to change from any one of the new scales into any other one, as for example, from the scale of 5 to that of 7, we must first change from the quinary to the denary, and then from the denary to the septenary.

Ex. Change 34201 from the quinary scale to the septenary.

First, converting $(34201)_5$ into the denary scale, I have

$$\begin{aligned}(34201)_5 &= 3\times 5^4+4\times 5^3+2\times 5^2+0\times 5+1\\ &= 3\times 625+4\times 125+2\times 25+1\\ &= 1875+500+50+1\\ &= (2426)_{10} \qquad\qquad (\text{IV.})\end{aligned}$$

$$\begin{array}{r} 7)\underline{2426}\\ 7)\underline{346} - 4\\ 7)\underline{49} - 3\\ 7)\underline{7} - 0\\ 7)\underline{1} - 0\\ \underline{\underline{0}} - 1\end{array}$$

Now, converting 2426 into the septenary scale, according to (200), as in the margin, we have as the new digits, 1, 0, 0, 3, 4, or $(2426)_{10} = (10034)_7$. And this equality may be proved by the reconversion of $(10034)_7$ into the denary scale; thus

$$\begin{aligned}(10034)_7 &= 1\times 7^4+0\times 7^3+0\times 7^2+3\times 7+4\\ &= 2401+21+4\\ &= (2426)_{10}.\end{aligned}$$

Hence, we have $(34201)_5 = (2426)_{10} = (10034)_7$.

Exs. 66.

1. Change 345 from the common or denary scale to the binary.
2. „ 10101 from the binary to the denary.
3. „ 1375 „ denary „ ternary.
4. „ $89t4$ „ undenary „ quinary.
5. „ 14328 „ nonary „ septenary.
6. „ 7854 „ denary „ duodenary.
7. „ $3t45$ „ undenary „ binary.
8. „ 11111 „ binary „ denary.
9. „ 4444 „ duodenary „ quinary.
10. „ 21021 „ ternary „ senary.
11. Add together in the senary scale, 25341, 5423, 4021, 13450.
12. Find in the duodenary scale the value of $89t36e - 4567t$.
13. Express the result of $187t56 \times t4789$ in the undenary scale.
14. Find the quotient of $18763 \div 456$ in the nonary scale.
15. Find the area of a floor 15 ft. 11 in. by 7 ft. 9 in., by multiplying in the duodenary scale.

202. It is a very good exercise for pupils to be able, without using any specific rules, to work examples which are generally wrought by rules; or, as it is sometimes expressed, to work them according to the principles of

common sense, of course employing the four Simple and Compound Rules, and Reduction.

The most favourable examples for this exercise are those generally wrought by the Rules given in Simple and Compound Proportion, Fractions, and Practice.

I will work one or two of each, to illustrate the method intended.

Ex. I. If 9 men earn £15 10s. in 3 months, or 13 weeks, what does each man earn per week?

If in 13 weeks 9 men earn 310s.

then, in 13 weeks 1 man earns $\frac{310}{9}$s.

or in 1 week $\frac{1}{13}$th of $\frac{310}{9}$s.

or $\frac{310}{117}$s. or 2s. $7\frac{31}{39}$d.

Ex. II. If £240 be paid for bread sufficient to serve 49 persons for 18 months, when wheat is 48s. per quarter; how long will £235 find bread for 92 persons, when wheat is 56s. per quarter?

If £240 will serve 49 persons for 18 months,

£1 " 49 persons for $\frac{18}{240}$ months,

or " 1 person for 49 times $\frac{18}{240}$ months.

and since 48s. = $\frac{48}{20}$£,

$\therefore$ 48s., or $\frac{48}{20}$ £ will serve 1 person for $\frac{48}{20}$ of $\frac{49 \times 18}{24}$ months.

Again, wheat, at the new price, 56s., will find as much as 48s. did before.

$\therefore$ 56s. will now find 1 person for $\frac{48}{20}$ of $\frac{49 \times 18}{24}$ months.

and since £1 = $\frac{20}{56}$ of 56s.

£1 will find 1 person for $\frac{\cancel{20}}{56}$ of $\frac{48}{\cancel{20}}$ of $\frac{49 \times 18}{240}$ months.

or " 92 persons for $\frac{1}{92}$th of $\frac{48}{56} \times \frac{49 \times 18}{240}$ months.

hence, £235 will find 92 persons for

$$\frac{235}{92} \times \frac{48}{56} \times \frac{49 \times 18}{240} \text{ months.}$$

This question worked as a Compound Proportion Example, according to (140), would furnish the following statement.

$$\left.\begin{array}{l} £240 \\ 92 \text{ men} \\ 56s. \end{array}\right\} : \left.\begin{array}{l} £235 \\ 49 \text{ men} \\ 48s. \end{array}\right\} :: 18 \text{ months.}$$

and the fourth term would $= \dfrac{18 \times 235 \times 49 \times 48}{240 \times 92 \times 56}$ months,

the same as before.

Ex. III. Find the present worth of £169 18s. 4d., due 15 months hence, at 5 per cent.

Here £100 in 15 months at 5 per cent. produces £6 5s., or amounts to £106 5s.

i. e. the present worth of £106¼ is £100,

and ∴ of £1 is £ $\dfrac{100}{106\frac{1}{4}}$.

and ∴ of £169 18s. 4d., or of £169$\frac{11}{12}$, is $\dfrac{169\frac{11}{12} \times 100}{106\frac{1}{4}}$£.

the result which would be obtained by a statement, such as in (149).

Ex. IV. If 7 men can do a piece of work in 11½ days, in what time will 8 men and 7 boys do the same, reckoning a boy's labour worth $\frac{5}{8}$ that of a man?

$$7 \text{ boys} = \frac{5}{8} \text{ of } 7 \text{ men}$$

$$8 \text{ men and } 7 \text{ boys} = \left(8 + \frac{5}{8} \text{ of } 7\right) \text{ men,}$$

$$= \left(8 + \frac{35}{8}\right) \text{men} = (8 + 4\tfrac{3}{8}) \text{ men} = 11\tfrac{3}{8} \text{ men.}$$

now if 7 men do the work in 11½ days,

1 man will do it in 11½ × 7 days,

∴ 11⅜ men „ in $\dfrac{11\frac{1}{2} \times 7}{11\frac{3}{8}}$ days,

or, in $\dfrac{\frac{23}{2} \times 7}{\frac{91}{8}}$ days,

or, in $\dfrac{23}{\cancel{2}} \times \dfrac{\cancel{7}}{1} \times \dfrac{\overset{4}{\cancel{8}}}{\underset{13}{\cancel{91}}}$ days, or $\dfrac{92}{13}$ days.

or $7\frac{1}{13}$ days.

Exs. 67.

MISCELLANEOUS EXAMPLES.

1. What will be the net value of a legacy of £333 6s. 8d., after paying duty at the rate of 3 per cent.?

2. Two clocks point to 2 at the same instant; one loses 7 seconds, and the other gains 8 seconds in 24 hours; when will one be half an hour before the other; and what time will each clock then shew?

3. A bankrupt owes 3 creditors £10,000, 10,000 guineas, and 10,000 shillings respectively; but his property is worth only £7000: find how much in the pound he will pay, and how much each creditor will receive?

4. If 100 articles are bought at 3 a penny, and 100 more at 2 a penny: at what price must they be retailed so as to gain 25 per cent.?

5. A vessel containing 384 gallons is emptied by 3 taps; the first and second together empty it in 32 minutes; the first and third in 24 minutes; and the second and third in 16 minutes; how many gallons will each tap discharge in a minute?

6. How many square feet of board will be required to make a rectangular box, of which the length, breadth, and depth are $3\frac{1}{2}$ ft., $2\frac{1}{4}$ ft., and 1 ft. $2\frac{1}{2}$ in. respectively?

7. If a room be 32 ft. long, 26 ft. wide, and 14 ft. high, what will be the expense of papering it at 3s. per square yard, allowance being made for three windows, each 10 ft. by $6\frac{1}{4}$ ft., and two fire-places, each 6 ft. by $9\frac{1}{2}$ ft.?

8. What sum of money must be paid down, in order to receive £360 10s. 2 yrs. hence, allowing $3\frac{1}{2}$ per cent. Compound Interest?

9. Find the simple fraction which expresses the value of $7+\sqrt{6\frac{1}{4}}$, when divided by $6\frac{1}{2}$ times $(3+\sqrt[3]{3\frac{3}{8}})$.

10. How many cubes, whose edges are $\frac{3}{4}$ in. long, can be contained in a box, of which the base is 18 sq. inches, and height $7\frac{1}{2}$ inches?

11. The volumes of spheres are in proportion to the cubes of their radii: the radii of two spheres are in the ratio of 4 to 5; and the weights of equal portions of the smaller and larger spheres are as 12 : 7; given that the weight of the smaller sphere is 256 lbs., find the weight of the larger one.

12. At what time between one and two o'clock will the hour and minute hands of a watch make an angle of 60 degrees with each other?

13. If a pound Troy of English standard gold $\frac{11}{12}$ fine be worth £46 12s. 6d., what is the value of a coin weighing 7 dwts. 11 grs., in which 924 parts in 1000 are pure gold?

14. The wheels of a cart are $2\frac{1}{4}$ yards asunder, and the inner wheel describes the circumference of a circle of radius 20 yds.: find the difference of the paths of the wheels, having it given that the circumference of a circle = 3·1416 times its diameter.

15. Two men are walking in the same direction, the distance between them at starting being 100 yds.; the first walks 45, and the second 49 yds. in 50 steps; how many steps will have been taken when they are together?

16. How much paper, $\frac{3}{4}$ yd. wide, will be sufficient to paper a room 22 feet 5 inches long, 12 feet 1 inch broad, and 11 feet 3 inches high? and how much will it cost at $4\frac{1}{2}$d. per yard?

17. A river 30 feet deep, and 200 yds. wide, is flowing at the rate of 4 miles an hour; find how many cubic feet of water run into the sea per minute; also the number of tons, supposing a cubic foot of water to weigh 1000 ounces.

18. A clock gains $3\frac{1}{4}$ minutes per day; how should its hands be placed at noon, that it may point out the true time at $7\frac{1}{4}$ in the evening?

19. A person performs $\frac{2}{7}$ths of a piece of work in 13 days; he then receives the assistance of another person, and the two together finish it in 6 days; in what time could each do the whole work by himself?

20. A tradesman buys goods for £189 15s. 6d., and sells them in 6 months for £253 0s. 8d. ready money: how much is that per cent. per annum profit?

21. If the value of £1 sterling varies from 25·15 francs to 26·75 francs, what is the variation in value of 100 guineas?

22. A person receives his rent, and after paying an income tax of 7d. in the pound has £553 7s. 6d. left; what did he receive?

23. A field of $7\frac{1}{4}$ acres is planted with potatoes in rows; the distance between each row is 15 inches; how many yards of potatoes are there in the field?

24. Three masons, A, B, C, are to build a wall; A and B could together build it in 12 days, B and C in 20, and A and C in 15; in wha time will they build it when they all work together?

25. If 3 miles, 4 furlongs, 93 yards be run in 6 minutes 4 second how much is that short of the rate of a mile per minute?

26. Multiply $\dfrac{2\frac{3}{8}}{4\frac{3}{4}}$ by $\dfrac{2\frac{1}{5}\frac{1}{2}}{13\frac{5}{18}}$, and divide $\sqrt{2\frac{1}{4}}$ by $\sqrt[3]{3\frac{3}{8}}$.

27. The comparative weights of coal and water are as 1·12 and also a cubic foot of water weighs 1000 oz.; find the edge of a cul block of coal which weighs 2000 tons.

28. The discount on £500 due 4 years hence is 250 marks? fin rate of interest.

29. Shew what factor is wanting in the number 32 to make it a j

30. Required the least number which multiplied by 64, will make it a perfect 5th power.

31. *A* sets out from Cambridge to London ($51\frac{1}{4}$ miles) at the rate of 8 miles an hour, and *B* sets out at the same time from London to Cambridge at the rate of $9\frac{1}{2}$ miles an hour, at what distance from each place will they meet?

32. A tradesman marks his goods with two prices; one for ready money and the other for credit of 6 months; what fixed proportion ought the two prices to bear to each other, allowing 5 per cent. Simple Interest?

33 What must be the least multiplier of the number 225, so that the product may be a perfect cube?

34. If the ratio of the diameter of a circle to its circumference be 113 : 355; and if the length of $\frac{1}{360}$th part of the earth's circumference be $69\frac{1}{12}$ miles, what is the earth's diameter?

35. How much per cent. is 14s. 6d. of £3 10s.?

36. *A* and *B* together can do a piece of work in 30 days; *B* by himself can perform the same in 70 days: in what time could *A* finish it by himself; and how much more of the work does *A* do than *B*?

37. The sun's longitude is increased by 360 degrees in 365 d. 5 h. 48 m.: what is his average daily motion?

38. Simplify the following expression, leaving one surd in the numerator of the resulting fraction:—

$$\sqrt{6^2 - 5^2} \times \frac{5}{\sqrt{6^2 + 5^2}} + \sqrt{6^2 + 5^2} \times \frac{5}{\sqrt{6^2 - 5^2}}$$

39. *A* and *B* can do a piece of work tokether in 30 days; *A* does $\frac{1}{4}$ more than *B*; in what time can they do it separately?

40. Find the square root of the sum of the squares of ·2, ·4, ·6, ·86.

41. If an acre of land be bought for 6d. per foot, at what price per yard must it be sold to gain £136 2s. 6d.?

42. From a rectangular plank 1 foot broad and 2 inches thick, what length must be cut off to be worth 25s., the value of the timber being 5s. per cubic foot?

43. Simplify the following expressions:—

$$1 + \cfrac{2}{3 + \cfrac{1}{4 + \cfrac{1}{5\frac{2}{3}}}} \qquad \frac{\sqrt{1+\frac{1}{3}} \div \sqrt{1-\frac{1}{5}}}{\sqrt{1+\frac{1}{3}} \times \sqrt{1-\frac{1}{5}}}$$

44. Find the value of $3 + \sqrt{10}$ to four places of decimals.

45. If the volume of a cylinder be obtained from this product, height × area of the base; find the area of the base of a cylinder whereof the volume is 1 cubic foot, and the height $7\frac{1}{2}$ inches.

46. Compare the volumes of two cylinders whose bases are in the ratio of 3 : 4, and altitudes as 5 : 6.

47. A person rents a piece of land for £120 a year. He lays out £625 in buying 50 bullocks. At the end of the year he sells them, having expended £12 10s. in labour. How much per head must he gain by them, in order to realize his rent and expenses, and 10 per cent. on his original outlay?

48. A cubical box contains 37 solid feet and 64 solid inches: find (1) the number of linear inches in the edge; (2) in the diagonal of each face; and (3) in the diagonal of the box.—*(See Appendix, Note to Art. Ratio.)*

49. A person finds that his net income, after deducting the income tax of 7d. in the pound, is £233: find the amount of income.

50. A pile of cannon shot has a rectangular base, the sides of which contain 7 and 6 shot respectively; find the number of shot in the whole pile.

51. What is the unit of measurement when a mile is 160 units? Determine that common unit which will express 12960 minutes and 20160 minutes in the smallest possible integral numbers.

52. The areas of circles are proportional to the squares of their radii; find the ratio of the areas of two circles which have 2 feet and $\frac{1}{2}$ an inch as the values of their respective radii.

53. Find the ratio of the diameters of two circles, such that the area of one may $= 5\frac{4}{9}$ times that of the other.

54. Two cubical boxes have edges respectively $\frac{3}{4}$ inch, and 2 ft. 3 in.; find the ratio (1) of their surfaces, and (2) of their volumes.

55. A clock has its face marked so as to shew 24 hours in a day; and on a certain evening half an hour after sunset it was set at 24 o'clock. The morning following it was 8 min. past 4 by a common clock when it was 4 minutes past 8 by this clock. Find the time of sunset the previous evening.

56. Given that the ratio of the circumference of a circle to the diameter $= 3{\cdot}1416 : 1$; also that the length of an arc of a circle opposite to any angle at the centre is proportional to the degrees, &c. in that angle: find the length of the radius of that circle, in which an arc of 4000 miles is opposite to an angle containing 8·58 seconds.

57. The space through which a body falls in vacuo near the surface of the earth is proportional to the square of the number of seconds in the time of falling: if a body fall through 16·1 feet in the first second, find how far it will fall in 8 seconds, and in the ninth.

58 The time of oscillation of the pendulum of a clock $= 3{\cdot}1416 \times \sqrt{\frac{39{\cdot}2}{32{\cdot}2}}$ seconds, where 39·2 inches is the length of the seconds pedulum; find the alteration in the time of oscillation when the pendulum is length-

ened $\frac{1}{20}$ of an inch. Find also how many seconds the clock will gain or lose in 24 hours.

59. Taking the same value as in the last question for the time of oscillation, where 32·2 is the measure of the force of the earth's attraction, find the alteration of that measure, if the length of the pendulum be increased by $\frac{1}{1000}$th part, and the time of oscillation be the same.

60. The difference between the year in the Julian calendar and the true year is ·007736 days; the Gregorian calendar corrects by omitting 3 days in 400 years: find how much error would have accumulated under that calendar, from A.D. 325 to A.D. 1848, and how soon the error will amount to a day.

61. In a block of wood, a hole is made 12 in. long and 1 sq. inch in section: the largest possible cylinder is placed in the hole; how much is unoccupied?—*(See question* 45.)

62. A cubical box, 1 inch high, is filled with water; 8 equal spheres of $\frac{1}{2}$ in. diam. are placed in it; what volume of water will remain,—having it given that the volume of a sphere $= \frac{4}{3} \times 3{\cdot}1416 \times$ the cube of the radius.

63. Will $\frac{21}{140}$ produce a circulating decimal?

64. A cube has an edge 2 ft. 6 in. long; find the ratio between the sum of the areas of the semicircles described on its edges, and the whole surface of the cube, having it given that the area of a circle $= 3{\cdot}1416 \times$ the square of the radius.

65. A wall is 15 ft. 8 in. long, and 11 ft. 6 in. broad, and has in it a door-way 6 ft. 3 in. by 2 ft. 4 in.; find the number of bricks of $165\frac{7}{8}$ solid inches contained in it, when the thickness is 11 inches.

66. If a sovereign weigh 5 dwts. $3\frac{1}{4}$ grs., 1 part out of 12 being copper, and the rest pure gold; find what fraction of a cubic inch the gold constitutes, having given that a cubic inch of water weighs 252·458 grains, and that gold is 19·362 times as heavy as water.

67. Reduce the following Arithmetical expression to its simplest form:

$$\left\{\left(\frac{9}{10}+2\tfrac{1}{4}\right)-\left(2\tfrac{1}{3}-1\tfrac{2}{3}\right)\right\} \times \left\{\left(5\tfrac{1}{4}+7\tfrac{3}{8}\right) \div 16\tfrac{1}{10}\right\}$$

68. A cistern has 3 pipes, *A*, *B*, and *C*; *A* and *B* can fill it in 3 and 4 hours respectively; and *C* can empty it in 1 hour; if these pipes be opened in order at 1, 2, and 3 o'clock, find when the cistern will be empty.

69. A person sells out of the $3\frac{1}{2}$ per cents at $93\frac{1}{2}$, as much stock as produces £9350; at what price must the 4 per cents be, so that the above sum when invested in them shall produce an increase of £10 income?

70. If $\frac{1}{5}$ of a sheep be worth $\frac{3}{4}$£, and $\frac{3}{7}$ of a sheep be worth $\frac{1}{12}$ of an ox, what sum must be given for 50 oxen?

71. A garden walk, 4 ft. wide, is carried round a circular plot 3 yds. in diameter; find the price of gravelling the walk per foot, if the whole cost be 6s.

APPENDIX.

FRACTIONAL QUOTIENT.

203. It was mentioned in (22) that the word quotient is not inapplicable, where one number is to be divided by another, and yet the division cannot be performed; thus in the division of 2 by 5, we said that the quotient was $\frac{2}{5}$.

This may be seen more clearly, if we remember that the word *quotient* means *how often* the divisor is contained in the dividend,—and if we observe the following illustration. Suppose that I have a rod five inches long, and with it I am to measure the depth of water in several full vessels, of which the depth is known to be 1, 2, 3, 4, ... inches. If now I ask *how many* times the length of this rod is contained in the depth of the water; the reply is, that in the first, one inch or one-fifth of the rod was immersed, *i.e.* the rod *went into* the depth one-fifth times; and therefore the genuine quotient of 1 when divided by 5 is one-fifth or $\frac{1}{5}$. Similarly, the quotient in the case of the second, third, &c. vessels, would be $\frac{2}{5}$, $\frac{3}{5}$, &c. And when I come to measure the depth of the five-inch vessel, the quotient ought to be $\frac{5}{5}$, or 1, *i. e.* the rod is just once immersed. And when I measure the seven inch depth, I find the quotient to be $\frac{7}{5}$ or $1\frac{2}{5}$; *i. e.* it takes one immersion of the *rod,* and $\frac{2}{5}$ of another, to measure the seven inches.

COMPOSITE DIVISOR.

204. In the operations of Reduction we frequently have to divide by composite divisors, which consist of the product of two factors, each not exceeding 12; and by dividing by these component parts, we can employ Short, instead of Long Division. For instance, in reducing grs. to dwts., I have to divide by 24: but since $24 = 6 \times 4$, it will be seen that if I divide by 6 and 4 successively, I obtain a correct result; but the main difficulty is to obtain the true remainder.

Ex. Convert 3887 grains into dwts.

$$24\left\{\begin{array}{l} 6)\ \underline{3887} \\ 4)\ \underline{647} \\ \quad \underline{\underline{161}} \end{array}\right. \begin{array}{l} \\ 5 \\ 3 \end{array}\!\!\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}23 \text{ grs.}$$

Dividing by 6, we have a quotient 647 and remainder 5. Now, dividing the grs. by 6, is equivalent to separating them into parcels, each containing 6 grs., and the remainder is of the same kind as the 3887, viz. grs.; *i.e.* 3887 grs. = 547 parcels of 6 grs. + 5 grs.

Again, since four parcels of 6 grs. each will make one parcel of 24 grs., or 1 dwt.; therefore, upon dividing the 647 by 4, the quotient will become dwts.; and the remainder, if any, will be parcels of 6 grs. each: the quotient is 161, and remainder 3. Hence the whole remainder = 3 parcels of 6 grs. + 5 grs., or = 3×6 grs. + 5 grs. = 18 grs. + 5 grs. = 23 grs. And if the former divisor had been any other number than 6—as 5, 8, or 10, the second remainder 3 would have represented 3 parcels of 5, 8, or 10 grs. each: hence the true remainder is found by multiplying the second remainder by the first divisor, and adding in the second remainder; of course, if the former remainder be wanting, the second remainder multiplied by the first divisor will be the whole remainder: and if the second remainder be wanting, the former remainder is the true one.

Again, working fractionally, we find that the first division gives as a fractional remainder $\frac{5}{24}$ dwt.; because the required quotient is to be in dwts., and the remaining 5 grs. are $\frac{5}{24}$ of 1 dwt. The second remainder is $\frac{3}{4}$ dwt., because the 161 consists of dwts.; hence the whole remainder is $\frac{5}{24}$ dwts. + $\frac{3}{4}$ dwts. = $\frac{5+18}{24}$ dwts. = $\frac{23}{24}$ dwts.; and it will be observed that

the second numerator 3, which is the second remainder in the former method, was multiplied by 6 before it could be added to 5, the former numerator, *i. e.* the former remainder: hence by both processes the second remainder 3 has been multiplied by 6, so that the two methods of obtaining the remainder coincide in their operation as well as in their result.

205. We may also explain the correctness of the usual method of dividing by 20, 30, &c. ...... or any number formed by the multiplication of 10 and any number not exceeding twelve.

Ex. To divide 3275 by 20.

$$20\left\{\begin{array}{r} 10)\ \underline{3275} \\ 2)\ \underline{\ 327}\ \ 5 \\ \underline{\underline{163}}\ \ 1 \end{array}\right.\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}15$$

Performing the division by the factors 10 and 2, as in the last article, we observe that the division by 10 gives the same result as merely cutting off the last figure as a remainder, and considering all the other numbers as the quotient; and that if there be a remainder after the second division, a figure 1 of such remainder represents not 1 but 10: hence the whole remainder is $10 + 5 = 15$.

And the work may be written thus, $\begin{array}{l} 2{,}0)\ \underline{327{,}5} \\ \underline{\underline{\ \ \ \ 163\ ,,\ 15}} \end{array}$, or $163\frac{15}{20} = 163\frac{3}{4}$.

RATIO.

206. It was observed in (174) that quantities which cannot be accurately represented by numbers, are said to be incommensurable. And since the Exs. in Ratio and Proportion, which we have considered, have always involved only commensurable quantities, it might be thought that the application of the principles of Ratio and Proportion was limited to such quantities.

But by observing (66), from whence our definition of Ratio was taken, we learn that a Ratio can exist between any two quantities whose magnitude can be represented, as A and B are, *i. e.* geometrically: and since it can be

shown* that magnitudes which cannot be represented accurately in numbers, can yet be correctly represented geometrically; therefore a ratio can exist between two or more quantities, even if one or all of them be incommensurable, though the value of that ratio cannot be accurately represented in numbers. Thus, in Fig. 1, page 171, the ratio of the diagonal of a square inch to its side is represented by the fraction $\frac{a\text{———}c}{a\text{———}b}$; or arithmetically, $\frac{1{\cdot}4142\ldots.}{1}$ or 1·4142.....

CIRCULATING DECIMALS.

207. Sometimes a decimal of very long period may be easily carried out to many places, without performing the division throughout, as in the following Ex.

To reduce $\frac{1}{19}$ to a decimal.

```
19) 1·00 (·05263
     95
     50
     38
    120
    114
     60
     57
      3
```

By division we have

$$\frac{1}{19} = {\cdot}05263\frac{3}{19} \qquad \text{(H)}$$

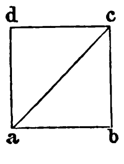
therefore, multiplying both sides by 3,

$$\frac{3}{19} = {\cdot}15789\,\frac{9}{19}$$

and substituting for $\frac{3}{19}$ in (H), we have

$$\frac{1}{19} = {\cdot}05263157789\frac{9}{19};$$

therefore $\frac{9}{19} = {\cdot}4736842101\,\frac{81}{19} = {\cdot}4736842105\frac{5}{19};$



* Some knowledge of Geometry is necessary to enable us to find the length of the line $a\ c$.

Since $a\ b\ c$ is one angle of a *rectangular* figure, or a *right angle*, we learn from Euclid, I. 47, that the square of which $a\ c$ is the side = the sum of the squares of which $a\ b$ and $b\ c$ are the sides. Now, in Fig. 1, p. 171, we took $a\ b = 1$, and $a\ c = 1$; therefore the sum of the squares of $a\ b$ and $a\ c = 1^2 + 1^2 = 2$; therefore the square of $a\ c = 2$, or $a\ c$ itself $= \sqrt{2}$.

ıd hence $\frac{1}{19} = \cdot 05263157894736842105\frac{5}{19}$; and by continuing this rocess, it is plain that we double at every step the number of figures ›reviously obtained. This decimal, it will be seen, circulates after the eighteenth figure; so that $\frac{1}{19} = \cdot\dot{0}521631578947364 2\dot{1}$.

PROPORTION.

208. The pupil who is acquainted with Book VI. of Euclid's Elements of Geometry, will know that if two

FIG. 9.

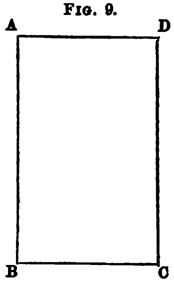
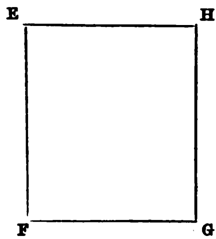


FIG. 10.



rectangular figures, AC, EG, be equal, the four sides whic' contain a pair of the equal angles, ABC, EFG, are propo: tional in the following order,—AB, EF, FG, BC; so th

$$AB : EF :: FG : BC \qquad (J).$$

We can show that this proportion is such as would obtained from the statement of a question in Rul€ Three, in which two surfaces AC, EG, are required tc equal.

Ex. How wide a piece of cloth, 15 feet long, will cover a floor 1 6 inches long and 10 feet wide?

Let AB be the length = 15 feet; BC the breadth, which is ye found; EF = 13 ft. 6 in.; FG = 10 ft.; then by the usual stateme writing BC as the fourth term, we have

15 ft. : 13 ft. 6 in. :: 10 ft. : BC;

and this will be found to be the same as (J).

In the language of Geometry, we say that the sides about the equal angles *ABC*, *EFG*, are *reciprocally* proportional.

209. The solution of such a question as the following is worth notice.

The shadow of a steeple is 105 yards long, and that of a stick 4½ feet long is 15 yards: find the height of the steeple.

FIG. 11.

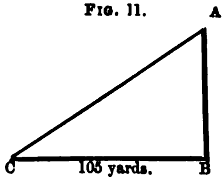
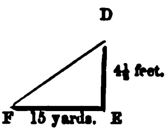


FIG. 12.



Let *AB*, *BC*, represent the steeple and its shadow; and let *DE*, *EF*, represent the stick and its shadow: I have to find the length of *AB*.

Now, most pupils, seeing this question under the head of Rule of Three, would immediately take it for granted that the three terms given would form a statement; *i. e.* that the ratio between the lengths of the two shadows is equal to that of the steeple and stick by which those shadows are cast. This is quite true; and my object in explaining this sum is merely to show what authority we have for believing that these two ratios are equal. Join *AC*, *DF*,—these lines *AC*, *DF*, will represent the direction of rays of light from the sun, and being from the same distant body are considered parallel: hence, since *AB*, *DE*, are parallel, as are also *FE*, *CB*; the triangles, *ABC*, *DEF*, are said to be similar: and from a geometrical property of such triangles, we have the following:

$$EF : BC :: DE : AB;$$

or, substituting the value of *EF*, *BC*, *DE*, which are given

by the question, we have

$$15 \text{ yards} : 105 \text{ yards} :: 4\tfrac{1}{2} \text{ feet} : AB;$$

and therefore $AB = \dfrac{105 \text{ yds.} \times 4\frac{1}{2} \text{ ft.}}{15 \text{ yards}} = 31\tfrac{1}{2}$ feet.

The following Ex. will show that it is not always safe to assume that a question which is apparently a Rule of Three Ex. will at once furnish a statement; *i.e.* that two quantities which appear to be connected, as in ordinary Exs., are really proportional to one another, as in (75) and (76).

Ex. If a ball falling from rest drop through a space of $64\frac{2}{5}$ feet in two seconds, through what distance will it have fallen in three seconds?

Now, if it were true that the distance fallen in any time were *directly proportional* to the time, the statement would be

$$2 \text{ seconds} : 3 \text{ seconds} :: 64\tfrac{2}{5} \text{ feet},$$

and the fourth term would be $96\frac{3}{5}$ feet. But a knowledge of the laws of mechanics corrects this supposition, and teaches us that the distance fallen in any time is not directly proportional to the number of seconds, but to the *square* of that number; and that in the above statement I ought to have put 2^2 and 3^2, or 4 and 9, instead of 2 and 3: hence the statement

$$4 : 9 :: 64\tfrac{2}{5} \text{ feet};$$

and the distance fallen in three seconds $= 144\frac{9}{10}$ feet.

MISCELLANEOUS QUESTIONS.

ON FRACTIONS AND THE PRINCIPLES OF PROPORTION.

1. Explain the terms "Common Measure," and "Common Multiple."

2. Shew how to find the Least Common Multiple of any set of whole numbers. Take any numbers as an Example.

3. When are numbers said to be prime to one another? Give an example of three numbers which are prime to one another, but none of them primes.

4. What do you mean by prime *factors?* Resolve 1512 into its prime factors.

5. Shew what is the G. C. M. of 27, 144, 96, by breaking them into their prime factors.

6. Explain the terms *proper* and *improper* fractions. Give an Ex. of each.

7. State the two modes of multiplying fractions by whole numbers. Take an Ex., and shew that the modes are true.

8. State also the two modes of *dividing* fractions by whole numbers, and prove their correctness in any example.

9. What do you mean by reducing a fraction to lowest terms? Shew the correctness of the process employed.

10. Taking any Ex. of the above reduction, draw a figure which shall enable us to see by inspection that the fraction when so reduced remains unaltered *in value.*

11. For what purpose do we require the L. C. M. of any numbers?

12. Shew how to reduce improper fractions to mixed numbers, and the contrary; giving an example of each kind of fractions.

13. What must be done with fractions, before they can be added or subtracted? Take as Exs. $\frac{3}{4}+\frac{5}{6}$; $\frac{7}{8}-\frac{3}{5}$.

14. How do you compare two proper fractions, at sight, so as to ascertain which is the larger? Ex. Compare $\frac{7}{11}$ and $\frac{9}{13}$.

15. Shew that $\frac{5}{7}\times\frac{2}{3}=\frac{10}{21}$; and that $\frac{3}{5}$ of $\frac{2}{7}=\frac{6}{35}$.

16. Hence shew that *of* and the sign ($\times$) placed between two fractions have the same meaning.

17. Shew that the common rule for division of fractions, viz. "Invert the divisor, and proceed as in multiplication," is correct. Ex. Shew that $\frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \times \frac{7}{2}$.

18. What do you understand by the brackets in the following Ex. $(\frac{3}{5}+\frac{2}{7}) \times (\frac{2}{3}+\frac{3}{4})$?

19. Explain the process of exhibiting any fractional quantity in positive terms. Ex. Express $\frac{7}{8}$ of 27s. in shillings, pence, and fractional parts of a penny.

20. What is the meaning of the term RATIO? How do you express the ratio of 4 to 5, and of 5 to 4?

21. Give distinctive names to the ratios 2 : 5; 3 : 3; 5 : 2.

22. Exhibit the ratio of 3s. to 7½d. as an abstract number.

23. What is proportion? The test of proportionality among 4 quantities is whether or no the product of the extremes equals the product of the means. Prove this.

24. Explain the reason of the following operations in Rule of Three; 1st. The reduction of the 1st and 2nd terms to the same name: 2nd. The multiplying of the 2nd and 3rd terms together and dividing by the 1st.

25. Explain how the term *proportional* is used to embrace two kinds of proportion. Give an Ex. of each.

26. Give a description of the mode of working the following questions:
 (1) Reduce 3s. 6d. to the fraction of £1.
 (2) What fraction of a half guinea is equivalent to a moidore?

27. Work the following Ex., and be particular in forming a correct fractional quotient in the pence. Ex. (£8 17s. 9¼d.) $\div$ 6⅞d.

28. Shew how to compare fractional quantities: Ex. Compare $\frac{3}{5}$, $1\frac{2}{7}$, $\frac{5}{9}$.

29. Compare also $\frac{5}{7}$£, $\frac{7}{9}$ of a guinea, and $\frac{5}{8}$ of 16s. 8d.

30. If $\frac{7}{15}$ths of a piece of work were done in 1 hour, how soon would all the work be done? What relation is there between the work done in 1 hour, and the time of doing all the work?

ON DECIMALS.

1. Explain the term "Decimal Fractions;" and shew the princi upon which we write tenths, hundredths, thousandths, &c. decimally.

2. Shew how to convert a *vulgar* fraction into a *decimal* fract written as a vulgar fraction.

3. Explain what is meant by a power of any number; and write divided by the 4th power of 10, (1) as a vulgar frac^n; (2) as a decin

4. State the method of converting into a decimal, a fraction contains a power of 10 as a denominator.

5. How do you multiply a decimal by a power of 10? Ex. $3{\cdot}75 \times 10^4$.

6. How do you divide a decimal by a power of 10? Ex. $18{\cdot}76 \div 10^5$.

7. State what fractions produce *terminating* decimals, and what produce *non-terminating*. Explain the reason.

8. If the den^r of a fr^n, in its lowest terms, which produces a circulating decimal be known, what may be known concerning the length of the period of the decimal? Illustrate by the Ex. $\frac{6}{7}$.

9. Shew how to convert terminating decimals into vulgar fractions. Ex. ·0605.

10. Explain fully the mode of converting non-terminating decimals into vulgar fractions. Take as Exs. $\cdot\dot{3}4\dot{3}$, $\cdot 36\dot{7}2\dot{5}$.

11. State the mode of adding together decimals—(1) terminating, (2) non-terminating.

12. State the mode for subtraction.

13. Shew the truth of the common rule for the multiplication of decimals. Ex. Find the value of $1{\cdot}75 \times {\cdot}037$.

14. Explain the mode of performing Long Division in Decimals; and shew what varieties may occur in fixing the decimal point in the quotient. Give Exs. in illustration.

15. Shew how to perform the operations of multiplication and division of circulating decimals.

16. Find the value of $3{\cdot}756 \times 21{\cdot}9875$, correct to 2 places of decimals: and explain the principle and mode of working by the contracted form of multiplication in decimals.

17. Find the product of 13·0586 by 12·758, without either assuming the rule for multiplication of decimals, or converting them to vulgar fractions.

18. Give Exs. of reduction of decimals,
   (1) From a decimal of £1 into positive terms, as shillings, pence, &c.
   (2) From a quantity involving pounds, shillings, and pence, to the decimal of a moidore. Work both your Exs.

ON PRACTICE.

1. What do you mean by the term "aliquot parts"? Give Exs.

2. What is the highest aliquot part of £1? What of 1s.?

3. Write out tables of aliquot parts. (1) Of 1s. (2) Of £1.

4. Find by Practice the cost of 7108 lbs. at $9\frac{3}{4}$d. Explain all the work.

5. Write down one of every variety of Examples in Practice.

6. Explain the mode of finding the cost of $5605\frac{3}{8}$ articles at any price.

7. Shew also how to find the value of 1105 yds. at 7s. $6\frac{9}{10}$d.

8. Give an Ex. in Practice in which the usual mode of taking aliquot parts of the highest denomination in money cannot be employed. Shew how the Ex. is to be worked.

ON PROPORTION.

1. How many terms are generally found in a proportion?

2. What is the general object of a question in Rule of Three?

3. In stating a Rule of Three sum, which term do you write down first, and how do you select it?

4. When the statement is completed, describe the remaining steps of the work.

5. Write down the fractional form of the fourth term, in terms of the other three; and hence derive the rules for completing a sum after it is stated.

6. From the fractional form mentioned in the last question, shew that the fourth term will be of the same nature as the third term.

7. Give an example in Rule of Three involving four terms, of which one will not appear in the statement.

8. Shew how three terms are to be obtained from the following Ex. What is the length of a floor which is 16 feet broad, and equal in area to a floor 24 feet square?

9. What is the difference between Simple and Compound Proportion?

10. Shew that one statement may be made to produce the same result as two or more separate statements by Single Rule of Three. Construct an Example, and thereby prove the above.

ON THE APPLICATIONS OF PROPORTION.

1. Explain the terms *Principal*, *Rate*, *Interest*, *Amount*.

2. Distinguish between Simple and Compound Interest.

3. What other questions come properly under the head of Simpl Interest?

4. Shew that the following Rule for finding Simple Interest is tru "Multiply the sum given by the rate per cent., and divide by 100."

5. Shew how to find the amount of any sum of money when out *interest for* any number of years and days.

6. Write in a plain Rule of Three form the questions involved in the following Examples.

   (1) At what rate per cent. will £175 4s. 2d. amount to £196 4s. 8d. in 3 years?

   (2) In what time will £175 amount to £204 15s., at $4\frac{1}{4}$ p. c. per an.?

   (3) What sum will amount to £155 5s. in 3 years at 5 p. c. Simple Interest?

7. What do you mean by so many years' purchase being given for a piece of property? Construct and work an Ex. involving the above term.

8. Shew the difference between Interest and Discount.

9. Explain whether the debtor or creditor is benefited by counting discount as interest.

10. What is meant by discounting a bill?

11. State and explain the technical terms connected with such bills.

12. If a tradesman wishes to throw off any required discount off the invoice price of any article, as 20, 30, &c. per cent., shew what proportion of the net value must be added, so as to give a proper gross price.

   Ex. I. Find the invoice price of an article worth 21s., so as to allow of a discount of 15 per cent.

   Ex. II. If $\frac{3}{5}$ of a net price be put on, to make the gross price, what discount is it intended to throw off?

13. Explain the objects proposed in questions under the head of "Profit and Loss."

14. How do you in all examples introduce the condition of losing 10 per cent., or of gaining 10 per cent?

15. Write down a rule for working examples in Fellowship or Partnership.

   (1) When the time is the same.

   (2) When the time is different.

16. Shew how to divide any sum of money into parts proportional to any given numbers. Ex. Divide £325 into shares proportional to the numbers 2, 1, $\frac{1}{2}$.

17. Construct an Ex. concerning the mixing together of articles of the same species, but of different values; and find the value of any quantity of the mixture.

18. State and explain the terms used in the buying and selling of Stocks.

19. Place in a Rule of Three form the following questions.

   (1) What must be given for £2520 stock at $87\frac{7}{8}$, and 2s. 6d. per cent. commission?

   (2) What per centage will be obtained by investing in the 3 per cents at $96\frac{1}{8}$, allowing $\frac{1}{8}$ per cent. for commission?

   (3) How much stock can be bought for £1450, when the price $104\frac{1}{2}$, and commission 2s. 6d. per cent?

20. What are the principal questions that arise under the head of EXCHANGE?

21. Explain the terms *Par of Exchange*, *Course of Exchange*, and *Simple* and *Compound Arbitration*.

22. Explain the mode of payment for imports and exports between America and England, by means of Bills of Exchange.

23. What is meant by the Chain Rule? Construct an Ex. which will exemplify its use, and work it both with and without that rule.

24. What is the object of the Rule termed Equation of Payments? Explain the opposite views adopted in solving questions under this head.

25. Work the following question without using any statement:

If 12 yards of cloth, 3 qrs. wide, cost £19, what will be the cost of 8 yds. 5 qrs. wide?

AREA AND VOLUME.

1. Describe the several meanings of the term inch, foot, or yard, when used in 1, 2, and 3 dimensions.

2. Define the term *rectangular*.

3. Give the shapes and names of the areas formed by the following products:

(1) 1 foot × 1 foot.
(2) 1 inch × 1 foot.
(3) 1 inch × 1 inch.
(4) 1 inch × $\frac{1}{12}$ inch.
(5) $\frac{1}{12}$ inch × $\frac{1}{12}$ inch.

4. State the mode of finding the number of square feet, &c. in a rectangular area.

5. Shew how the Tables called "Square Measure," and "Solid Measure," have been formed.

6. Shew by actual computation of the area of a slab, 3 ft. 7 in. by 2 ft. 9 in., that the process given by Cross Multiplication is correct.

7. Give the shapes and names of the solids formed by the following products:

1 sq. foot × 1 linear inch.
1 sup. pr. × 1 linear inch.
1 sq. inch × 1 linear inch.

8. Shew by actual computation of the volume of a block, 4 ft. 6 in. long, 2 ft. 4 in. wide, and 1 ft. 9 in. thick, that the process given by Cross Multiplication is correct.

9. Find the value of the above volume by multiplying together the above dimensions, as fractional parts of a foot.

10. Explain the Gunter's Chain; and shew how to convert an area expressed in links into acres, &c.

11. What is meant by an incommensurable quantity? Give an example of such a quantity.

EXTRACTION OF ROOTS.

1. Explain the terms INVOLUTION and EVOLUTION.

2. What is meant by the following expression $\sqrt[5]{18}$?

3. Explain the term perfect square, cube, &c.

4. What will be the form of a square root of a number which is not a perfect square?

5. Give a name to such quantities as $\sqrt{2}$, $\sqrt[3]{5}$.

6. Explain the mode of *pointing*, previous to the extraction of the square root, both of whole numbers and decimals.

7. Shew how to find a multiplier which shall make any proposed number a perfect square, or a perfect cube:

   Ex. Find the multiplier which shall make 45, (1) a perfect square, (2) a perfect cube.

8. Write down the squares of all the digits.

9. Give a rule for the extraction of the square root.

10. Shew why the incomplete divisor does not always give a correct figure in the quotient.

11. How do you find any power of a fraction? How do you extract the root of a fractional quantity?

12. Give two methods of extracting the square root of a fractional quantity, of which the denominator is not a perfect square.

13. Give a rule for the extraction of the Cube Root.

14. Give a rule for the extraction of any root whatever.

15. Extract the square root of 576, and the cube root of 1728, in an algebraical form.

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